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A q-enumeration of lozenge tilings of a hexagon with three dents

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ABSTRACT

MacMahon's classical theorem on boxed plane partitions states that the generating function of the plane partitions fitting in an $a \times b \times c$ box is equal to

$$\frac{\mathrm{H}_q(a)\mathrm{H}_q(b)\mathrm{H}_q(c)\mathrm{H}_q(a+b+c)}{\mathrm{H}_q(a+b)\mathrm{H}_q(b+c)\mathrm{H}_q(c+a)},$$

where $\mathrm{H}_q(n) := [0]_q! \cdot [1]_q! \dots [n-1]_q!$ and $[n]_q! := \prod_{i=1}^{n}(1+q+q^2+\cdots+q^{i-1})$. By viewing a boxed plane partition as a lozenge tiling of a semi-regular hexagon, MacMahon's theorem yields a natural q-enumeration of lozenge tilings of the hexagon. However, such q-enumerations do *not* appear often in the domain of enumeration of lozenge tilings. In this paper, we consider a new q-enumeration of lozenge tilings of a hexagon with three bowtie-shaped regions removed from three non-consecutive sides.

The unweighted version of the result generalizes a problem posed by James Propp on enumeration of lozenge tilings of a hexagon of side-lengths $2n$, $2n+3$, $2n$, $2n+3$, $2n$, $2n+3$ (in cyclic order) with the central unit triangles on the $(2n+3)$-sides removed. Moreover, our result also implies a q-enumeration of boxed plane partitions with certain constraints.

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1. Introduction

A *plane partition* is a rectangular array of non-negative integers so that all rows are weakly decreasing from left to right and all columns are weakly decreasing from top to bottom. The plane partitions having a rows and b columns with entries at most c are usually identified with their 3-D interpretations — piles of unit cubes fitting in an $a \times b \times c$ box. (Such plane partitions are usually called *boxed plane partitions.*) The latter piles of unit cubes are in bijection with the lozenge tilings of the semi-regular hexagon $Hex(a, b, c)$ of side-lengths a, b, c, a, b, c (in clockwise order, starting from the northwest side) on the triangular lattice. Here, a *lozenge* is a union of any two unit equilateral triangles sharing an edge, and a *lozenge tiling* of a *region*[1] is a covering of the region by lozenges so that there are no gaps or overlaps. The *volume* (or the *norm*) of the plane partition π is defined to be the sum of all its entries, and denoted by $|\pi|$.

Let q be an indeterminate. The *q-integer* is defined by $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$. We also define the *q-factorial* $[n]_q! := [1]_q \cdot [2]_q \cdot [3]_q \ldots [n]_q$, and the *q-hyperfactorial* $\mathrm{H}_q(n) := [0]_q! \cdot [1]_q! \cdot [2]_q! \ldots [n-1]_q!$. MacMahon's classical theorem [29] states that

$$\sum_{\pi} q^{|\pi|} = \frac{\mathrm{H}_q(a)\mathrm{H}_q(b)\mathrm{H}_q(c)\mathrm{H}_q(a+b+c)}{\mathrm{H}_q(a+b)\mathrm{H}_q(b+c)\mathrm{H}_q(c+a)}, \tag{1.1}$$

where the sum on the left-hand side is taken over all plane partitions π fitting in an $a \times b \times c$ box.

The $q = 1$ specialization of MacMahon's theorem is equivalent to the fact that the number of lozenge tilings of the hexagon $Hex(a, b, c)$ is equal to

$$\frac{\mathrm{H}(a)\mathrm{H}(b)\mathrm{H}(c)\mathrm{H}(a+b+c)}{\mathrm{H}(a+b)\mathrm{H}(b+c)\mathrm{H}(c+a)}, \tag{1.2}$$

where $\mathrm{H}(n) = \mathrm{H}_1(n) = 0!1! \ldots (n-1)!$ is the ordinary hyperfactorial. This consequence of MacMahon's theorem inspired a large body of work, focusing on enumeration of lozenge tilings of hexagons with defects (see e.g. [2,3,8,7,9,11,13,6,17], or the references in [31,32] for more extensive lists). Put differently, MacMahon's theorem gives a q-enumeration of lozenge tilings of a semi-regular hexagon. However, such q-enumerations are *rare* in the domain of enumeration of lozenge tilings. Together with the related work [24], this paper presents such a rare q-enumeration.

In 1999, James Propp [31] published a list of 32 open problems in the field of enumeration of tilings (equivalently, perfect matchings). Problem 3 on this list asks for the number of lozenge of tilings of a hexagon of side-lengths[2] $2n+3, 2n, 2n+3, 2n, 2n+3, 2n,$

[1] The regions considered in our paper are always finite connected regions on the triangular lattice.
[2] From now on, we always list the side-lengths of a hexagon in clockwise order, starting from the northwest side.