

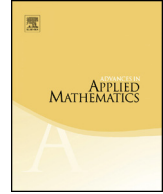


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A q -enumeration of lozenge tilings of a hexagon with three dents



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ARTICLE INFO

Article history:

Received 4 February 2016
Received in revised form 30 June 2016

Accepted 10 July 2016
Available online 27 July 2016

MSC:

05A15
05C30
05C70

Keywords:

Graphical condensation
Lozenge tilings
Perfect matchings
Plane partitions

ABSTRACT

MacMahon's classical theorem on boxed plane partitions states that the generating function of the plane partitions fitting in an $a \times b \times c$ box is equal to

$$\frac{H_q(a)H_q(b)H_q(c)H_q(a+b+c)}{H_q(a+b)H_q(b+c)H_q(c+a)},$$

where $H_q(n) := [0]_q! \cdot [1]_q! \cdots [n-1]_q!$ and $[n]_q! := \prod_{i=1}^n (1 + q + q^2 + \cdots + q^{i-1})$. By viewing a boxed plane partition as a lozenge tiling of a semi-regular hexagon, MacMahon's theorem yields a natural q -enumeration of lozenge tilings of the hexagon. However, such q -enumerations do *not* appear often in the domain of enumeration of lozenge tilings. In this paper, we consider a new q -enumeration of lozenge tilings of a hexagon with three bowtie-shaped regions removed from three non-consecutive sides.

The unweighted version of the result generalizes a problem posed by James Propp on enumeration of lozenge tilings of a hexagon of side-lengths $2n, 2n+3, 2n, 2n+3, 2n, 2n+3$ (in cyclic order) with the central unit triangles on the $(2n+3)$ -sides removed. Moreover, our result also implies a q -enumeration of boxed plane partitions with certain constraints.

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<http://dx.doi.org/10.1016/j.aam.2016.07.002>
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1. Introduction

A *plane partition* is a rectangular array of non-negative integers so that all rows are weakly decreasing from left to right and all columns are weakly decreasing from top to bottom. The plane partitions having a rows and b columns with entries at most c are usually identified with their 3-D interpretations — piles of unit cubes fitting in an $a \times b \times c$ box. (Such plane partitions are usually called *boxed plane partitions*.) The latter piles of unit cubes are in bijection with the lozenge tilings of the semi-regular hexagon $Hex(a, b, c)$ of side-lengths a, b, c, a, b, c (in clockwise order, starting from the northwest side) on the triangular lattice. Here, a *lozenge* is a union of any two unit equilateral triangles sharing an edge, and a *lozenge tiling* of a *region*¹ is a covering of the region by lozenges so that there are no gaps or overlaps. The *volume* (or the *norm*) of the plane partition π is defined to be the sum of all its entries, and denoted by $|\pi|$.

Let q be an indeterminate. The q -integer is defined by $[n]_q := 1 + q + q^2 + \dots + q^{n-1}$. We also define the q -factorial $[n]_q! := [1]_q \cdot [2]_q \cdot [3]_q \dots [n]_q$, and the q -hyperfactorial $H_q(n) := [0]_q! \cdot [1]_q! \cdot [2]_q! \dots [n-1]_q!$. MacMahon’s classical theorem [29] states that

$$\sum_{\pi} q^{|\pi|} = \frac{H_q(a)H_q(b)H_q(c)H_q(a+b+c)}{H_q(a+b)H_q(b+c)H_q(c+a)}, \tag{1.1}$$

where the sum on the left-hand side is taken over all plane partitions π fitting in an $a \times b \times c$ box.

The $q = 1$ specialization of MacMahon’s theorem is equivalent to the fact that the number of lozenge tilings of the hexagon $Hex(a, b, c)$ is equal to

$$\frac{H(a)H(b)H(c)H(a+b+c)}{H(a+b)H(b+c)H(c+a)}, \tag{1.2}$$

where $H(n) = H_1(n) = 0!1! \dots (n-1)!$ is the ordinary hyperfactorial. This consequence of MacMahon’s theorem inspired a large body of work, focusing on enumeration of lozenge tilings of hexagons with defects (see e.g. [2,3,8,7,9,11,13,6,17], or the references in [31,32] for more extensive lists). Put differently, MacMahon’s theorem gives a q -enumeration of lozenge tilings of a semi-regular hexagon. However, such q -enumerations are *rare* in the domain of enumeration of lozenge tilings. Together with the related work [24], this paper presents such a rare q -enumeration.

In 1999, James Propp [31] published a list of 32 open problems in the field of enumeration of tilings (equivalently, perfect matchings). Problem 3 on this list asks for the number of lozenge of tilings of a hexagon of side-lengths² $2n+3, 2n, 2n+3, 2n, 2n+3, 2n$,

¹ The regions considered in our paper are always finite connected regions on the triangular lattice.
² From now on, we always list the side-lengths of a hexagon in clockwise order, starting from the northwest side.

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