Advances in Applied Mathematics 82 (2017) 58–82

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Context-free grammars for permutations and increasing trees

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ARTICLE INFO

Article history:
Received 19 June 2016
Accepted 25 July 2016
Available online 16 August 2016

Dedicated to the Memory of Professor Dominique Dumont

MSC:
05A15
05A19

Keywords:
Context-free grammar
Eulerian grammar
Grammatical labeling
Increasing tree
Exterior peak of a permutation
Stirling permutation

ABSTRACT

We introduce the notion of a grammatical labeling to describe a recursive process of generating combinatorial objects based on a context-free grammar. By labeling the ascents and descents of Stirling permutations, we obtain a grammar for the second-order Eulerian polynomials. Using the grammar for 0-1-2 increasing trees given by Dumont, we obtain a grammatical derivation of the generating function of the André polynomials obtained by Foata and Schützenberger. We also find a grammar for the number $T(n,k)$ of permutations on $[n] = \{1, 2, \ldots, n\}$ with k exterior peaks. We demonstrate that Gessel's formula for the generating function of $T(n,k)$ can be deduced from this grammar. From a grammatical point of view, it is easily seen that the number of the permutations on $[n]$ with k exterior peaks equals the number of increasing trees on $\{0, 1, 2, \ldots, n\}$ with $2k + 1$ vertices of even degree. We present a combinatorial proof of this fact, which is in the spirit of the recursive construction of the correspondence between even increasing trees and up-down permutations, due to Kuznetsov, Pak and Postnikov.

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http://dx.doi.org/10.1016/j.aam.2016.07.003
0196-8858/

1. Introduction

In this paper, a *context-free grammar* G over a set $V = \{x, y, z, \ldots,\}$ of variables is a set of substitution rules replacing a variable in V by a Laurent polynomial of variables in V. For a context-free grammar G over V, the formal derivative D (introduced in [2]) with respect to G is defined as a linear operator acting on Laurent polynomials with variables in V such that each substitution rule is treated as the common differential rule that satisfies the following relations,

$$D(u+v) = D(u) + D(v),$$
$$D(uv) = D(u)v + uD(v).$$

For a constant c, we have $D(c) = 0$. Clearly, the Leibniz formula is also valid:

$$D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D^k(u) D^{n-k}(v).$$

Since $D(ww^{-1}) = 0$, we have

$$D(w^{-1}) = -\frac{D(w)}{w^2}.$$

A formal derivative D is also associated with an exponential generating function. For a Laurent polynomial w of variables in V, let

$$\mathrm{Gen}(w,t) = \sum_{n\geq 0} D^n(w) \frac{t^n}{n!}.$$

Then we have the following relations:

$$\mathrm{Gen}'(w,t) = \mathrm{Gen}(D(w),t), \tag{1.1}$$

$$\mathrm{Gen}(u+v,t) = \mathrm{Gen}(u,t) + \mathrm{Gen}(v,t), \tag{1.2}$$

$$\mathrm{Gen}(uv,t) = \mathrm{Gen}(u,t)\mathrm{Gen}(v,t), \tag{1.3}$$

where u, v and w are Laurent polynomials of variables in V and $\mathrm{Gen}'(w,t)$ stands for the derivative of $\mathrm{Gen}(w,t)$ with respect to t.

To illustrate the connection between context-free grammars and combinatorial structures, we recall the following grammar introduced by Dumont [4]:

$$G\colon \quad x \to xy, \quad y \to xy. \tag{1.4}$$

He showed that it generates the *Eulerian polynomials* $A_n(x)$. Let S_n denote the set of permutations on $[n] = \{1, 2, \ldots, n\}$. For a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, an index