Advances in Applied Mathematics 82 (2017) 83–101

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Complete Kneser transversals ☆

J. Chappelon [a,*], L. Martínez-Sandoval [b], L. Montejano [b], L.P. Montejano [a], J.L. Ramírez Alfonsín [a]

[a] Institut Montpelliérain Alexander Grothendieck, Université de Montpellier, Case Courrier 051, Place Eugène Bataillon, 34095 Montpellier Cedex 05, France
[b] Instituto de Matemáticas, Universidad Nacional Autónoma de México, Ciudad Universitaria, México D.F., 04510, Mexico

A R T I C L E I N F O

Article history:
Received 29 October 2015
Received in revised form 25 July 2016
Accepted 27 July 2016
Available online 18 August 2016

MSC:
52A35
05C15
52C40
52B40
52B99

Keywords:
Transversals
Kneser hypergraphs
Oriented matroids
Cyclic polytope

A B S T R A C T

Let $k, d, \lambda \geqslant 1$ be integers with $d \geqslant \lambda$. Let $m(k, d, \lambda)$ be the maximum positive integer n such that every set of n points (not necessarily in general position) in $\mathbb{R}^d$ has the property that the convex hulls of all k-sets have a common transversal $(d-\lambda)$-plane. It turns out that $m(k, d, \lambda)$ is strongly connected with other interesting problems, for instance, the chromatic number of Kneser hypergraphs and a discrete version of Rado's centerpoint theorem. In the same spirit, we introduce a natural discrete version m^* of m by considering the existence of *complete Kneser transversals*. We study the relation between them and give a number of lower and upper bounds of m^* as well as the exact value in some cases. The main ingredient for the proofs are Radon's partition theorem as well as oriented matroids tools. By studying the alternating oriented matroid we obtain the asymptotic behavior of the function m^* for the family of cyclic polytopes.

☆ Supported by the ECOS-Nord project M13M01, by CONACYT project 166306, by CONACYT grant 277462, by PAPIIT-UNAM project IN112614 and by Frenkel Foundation.
* Corresponding author.
E-mail addresses: jonathan.chappelon@umontpellier.fr (J. Chappelon), leomtz@im.unam.mx (L. Martínez-Sandoval), luis@math.unam.mx (L. Montejano), lpmontejano@gmail.com (L.P. Montejano), jorge.ramirez-alfonsin@umontpellier.fr (J.L. Ramírez Alfonsín).

http://dx.doi.org/10.1016/j.aam.2016.07.004
0196-8858/

1. Introduction

Let $k, d, \lambda \geqslant 1$ be integers with both $d, k \geqslant \lambda$. Consider the function $m(k, d, \lambda)$, defined to be the maximum positive integer n such that every set of n points (not necessarily in general position) in $\mathbb{R}^d$ has the property that the convex hulls of all k-sets have a common transversal $(d - \lambda)$-plane.

In [1], the following inequalities were obtained

$$d - \lambda + k + \left\lceil \frac{k}{\lambda} \right\rceil - 1 \leqslant m(k, d, \lambda) < d + 2(k - \lambda) + 1. \tag{1}$$

An interesting feature of the value of $m(k, d, \lambda)$ is its strong connection with the chromatic number of Kneser hypergraphs [4,5] as well as with the Rado's centerpoint theorem [7]. Indeed, for the former it is proved in [1] that

$$\text{if } \; m(k, d, \lambda) < n, \; \text{ then } \; d - \lambda + 1 < \chi\left(KG^{\lambda+1}(n, k)\right).$$

For the latter, recall that the well-known Rado's centerpoint theorem [7] states that if X is a bounded measurable set in $\mathbb{R}^d$ then there exists a point $x \in \mathbb{R}^d$ such that

$$\text{measure}\,(P \cap X) \geqslant \frac{\text{measure}\,(X)}{d + 1}$$

for each half-space P that contains x (see also [6] for the case $d = 2$).

Independently Bukh and Matoušek [3, Section 6] and Arocha, Bracho, Montejano and Ramírez-Alfonsín in [1] consider the following generalization of a discrete version of Rado's centerpoint theorem. Let $n, d, \lambda \geqslant 1$ be integers with $d \geqslant \lambda$ and let

$\tau(n, d, \lambda) \stackrel{\text{def}}{=}$ the maximum positive integer τ such that for any collection X of n points in $\mathbb{R}^d$, there is a $(d - \lambda)$-plane L_X such that any closed half-space H through L_X contains at least τ points.

By the hyperplane separation theorem we have that $n - \tau(n, d, \lambda) + 1$ is equal to the minimum positive integer k such that for any collection X of n points in $\mathbb{R}^d$ there is a common transversal $(d - \lambda)$-plane to the convex hulls of all k-sets, which is essentially $m(k, d, \lambda)$. Therefore, any improvement to the lower or upper bounds for $m(k, d, \lambda)$ will give important insight on the above interesting problem.

The purpose of this paper is to introduce and study a discrete version of the function $m(k, d, \lambda)$ which perhaps will allow us to improve lower or upper bounds for $m(k, d, \lambda)$. Let $k, d, \lambda \geqslant 1$ be integers with $k, d \geqslant \lambda$ and let $X \subset \mathbb{R}^d$ be a finite set. We call L a *Kneser transversal* of X if it is a $(d - \lambda)$-plane transversal to the convex hulls of all k-sets of X. If in addition L contains $(d - \lambda) + 1$ points of X, then L is a *complete Kneser* $(d - \lambda)$-*transversal*. Let us define