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An inequality for multiple convolutions with respect to Dirichlet probability measure

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ABSTRACT

A sharp multiple convolution inequality with respect to Dirichlet probability measure on the standard simplex is presented. Its discrete version in terms of the negative binomial coefficients is proved as well. The new bounds for the Dirichlet distribution and iterated convolutions are obtained as the consequences of the main result. Also some binomial, exponential, and generalized hypergeometric applications are discussed.

1. Introduction

In the paper, we focus on the sharp integral inequalities for weighted convolutions of complex-valued functions and their discrete versions. Being generated by the seminorm problem for pairs of power series these inequalities are used for many applications

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[10–12]. We extend the earlier results for weighted convolutions of two functions on a finite interval to a multiple convolution inequality with respect to Dirichlet probability measure on the standard simplex [16,20]. Also we obtain the discrete version of this convolution inequality as a binomial solution to the general seminorm problem for power series. The limit exponential version is presented as well. The new bounds for the Dirichlet distribution and iterated convolutions are obtained as the consequences of the main result. Also some generalized hypergeometric and binomial applications are discussed. All the cases of equality are described. The discrete and integro-polynomial predecessors of the obtained results are given in [8,10]. We mention that weighted multiple convolution inequalities are of increasing importance in various fields of mathematics, statistics, physics, and biology.

We use the standard hypergeometric notation (see, e.g., [6, Chs. 2, 6]). The Pochhammer symbol $(\alpha)_n$ stands for the shifted factorial:

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) \quad \text{for} \quad n \geq 1 \quad \text{and} \quad (\alpha)_0 = 1. \tag{1}$$

For any $j, k = 0, 1, \ldots,$ the generalized hypergeometric function ${}_jF_k$ is defined by the power series:

$${}_jF_k(\mu_1, \ldots, \mu_j; \nu_1, \ldots, \nu_k; z) = \sum_{n=0}^{\infty} \frac{\prod_{1\leq l\leq j}(\mu_l)_n}{\prod_{1\leq l\leq k}(\nu_l)_n} \cdot \frac{z^n}{n!}, \tag{2}$$

provided that $(\nu_l)_n \neq 0$ $(n \geq 1, l \leq k)$.

For a given power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\alpha > 0$, the α-convolution of f denoted by $f_{*\alpha}$ is defined by the formula:

$$f_{*\alpha}(z) = \sum_{n=0}^{\infty} \frac{a_n}{(\alpha)_n} z^n. \tag{3}$$

Some properties of functions (2) hold for arbitrary α-convolutions [10]. As usual, $B(\alpha_1, \ldots, \alpha_m)$ and $\Gamma(z)$ stand for the beta function of $m \geq 2$ variables and gamma function, respectively [4]:

$$\begin{aligned} B(\alpha_1, \ldots, \alpha_m) &= \int\limits_{\mathbb{E}_{m-1}} \left(\prod_{j=1}^{m-1} t_j^{\alpha_j - 1}\right)\left(1 - \sum_{j=1}^{m-1} t_j\right)^{\alpha_m - 1} dt_1 \cdots dt_{m-1} \\ &= \frac{\prod_{j=1}^{m}\Gamma(\alpha_j)}{\Gamma(\sum_{j=1}^{m}\alpha_j)} \quad (\alpha_j > 0, j \leq m), \ \Gamma(\alpha) = \int\limits_0^{\infty} t^{\alpha-1}e^{-t}dt, \end{aligned} \tag{4}$$

where $\mathbb{E}_{m-1}$ is the standard simplex in $\mathbb{R}^{m-1}$:

$$\mathbb{E}_{m-1} = \{(t_1, \ldots, t_{m-1}) \in \mathbb{R}^{m-1} : t_1, \ \ldots, t_{m-1} \geq 0; t_1 + \cdots + t_{m-1} \leq 1\}.$$