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Tropical differential equations

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A B S T R A C T

Tropical differential equations are introduced and an algorithm is designed which tests solvability of a system of tropical linear differential equations within the complexity polynomial in the size of the system and in the absolute values of its coefficients. Moreover, we show that there exists a minimal solution, and the algorithm constructs it (in case of solvability). This extends a similar complexity bound established for tropical linear systems. In case of tropical linear differential systems in one variable a polynomial complexity algorithm for testing its solvability is designed.

We prove also that the problem of solvability of a system of tropical non-linear differential equations in one variable is NP-hard, and this problem for arbitrary number of variables belongs to NP. Similar to tropical algebraic equations, a tropical differential equation expresses the (necessary) condition on the dominant term in the issue of solvability of a differential equation in power series.

0. Introduction

Tropical algebra deals with the tropical semi-rings $\mathbb{Z}_+$ of non-negative integers or $\mathbb{Z}_+ \cup \{\infty\}$ endowed with the operations $\{\min, +\}$, or with the tropical semi-fields $\mathbb{Z}$ or $\mathbb{Z} \cup \{\infty\}$ endowed with the operations $\{\min, +, -\}$ (see e.g. [7,8,10]).

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A *tropical linear differential equation* is a tropical linear polynomial of the form

$$\min_{i,j}\{a_i^{(j)} + x_i^{(j)},\, a\} \tag{1}$$

where the coefficients $a,\ a_i^{(j)} \in \mathbb{Z}_+ \cup \{\infty\}$, and a variable $x_i^{(j)}$ is treated as "j-th derivative of $x_i := x_i^{(0)}$".

For a subset $S_i \subset \mathbb{Z}_+$ we define the *valuation*

$$\mathrm{V}al_{S_i} : \mathbb{Z}_+ \to \mathbb{Z}_+ \cup \{\infty\}$$

of variable x_i as follows. For each $j \geq 0$ take the minimal $s \in S_i$ (provided that it does exist) such that $s \geq j$ and put $\mathrm{V}al_{S_i}(x_i^{(j)}) := \mathrm{V}al_{S_i}(j) := s - j$: in case when such s does not exist put $\mathrm{V}al_{S_i}(x_i^{(j)}) := \mathrm{V}al_{S_i}(j) := \infty$. We use a shorthand

$$\mathrm{V}al_{S_1,\dots,S_n} := \mathrm{V}al_{S_1} \times \cdots \times \mathrm{V}al_{S_n} : \mathbb{Z}_+^n \to (\mathbb{Z}_+ \cup \{\infty\})^n.$$

Observe that if X_i is a power series in t with the support $\{t^s,\ s \in S_i\}$ then $\mathrm{V}al_{S_i}(j)$ is the order $\mathrm{o}rd_t(X_i^{(j)})$ at zero of the j-th derivative $X_i^{(j)}$ w.r.t. t.

We say that $S_1, \dots, S_n$ is a *solution of the tropical linear differential equation (1)* if the minimum $\min_{i,j}\{a_i^{(j)} + \mathrm{V}al_{S_i}(j),\, a\}$ is attained at least twice or is infinite (as is customary in tropical mathematics [7,8]). The latter is a necessary condition of solvability in power series in t of a linear differential equation $\sum_{i,j} A_{i,j} \cdot X_i^{(j)} = A$ in several indeterminates $X_1, \dots, X_n$. Namely, the orders of power series coefficients equal $\mathrm{o}rd_t(A_{i,j}) = a_{i,j}$, $\mathrm{o}rd_t(A) = a$ and the support of X_i is S_i. More precisely, (1) expresses that at least two lowest terms of the expansion in power series of the differential equation have the same exponents. It is similar to that the tropical equations concern the lowest terms of the expansions in Puiseux series of algebraic equations which emerge following the Newton polygon method.

Thus, in the aspect of solvability tropical linear differential equations (as well as their non-linear counterpart, see sections 4, 5 below) w.r.t. linear differential equations play a role similar to the one played by tropical equations w.r.t. algebraic equations.

We study solvability of a system of tropical linear differential equations

$$\min_{i,j}\{a_{i,l}^{(j)} + x_i^{(j)},\, a_l\},\ 1 \leq l \leq k \tag{2}$$

where $1 \leq i \leq n$, $0 \leq j \leq r$ and for all finite coefficients $a_{i,l}^{(j)},\ a_l \in \mathbb{Z}$ we have $0 \leq a_{i,l}^{(j)}$, $a_l \leq M$. Thus, the *bit-size* of (2) is defined by $knr\log_2(M+2)$.

We say that a solution $T_1, \dots, T_n$ of (2) is *minimal* if the inequality $\mathrm{V}al_{T_1,\dots,T_n} \leq \mathrm{V}al_{S_1,\dots,S_n}$ holds pointwise for any solution $S_1, \dots, S_n$ of (2).

Note that (2) extends tropical linear systems when for all the occurring derivatives $x_i^{(j)}$ we have $j = 0$. Thus, the complexity bound of testing solvability of (2) in the next