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Variations of the Poincaré series for affine Weyl groups and q-analogues of Chebyshev polynomials

Eric Marberg [*,1], Graham White

Department of Mathematics, Stanford University, United States

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A B S T R A C T

Let (W, S) be a Coxeter system and write $P_W(q)$ for its Poincaré series. Lusztig has shown that the quotient $P_W(q^2)/P_W(q)$ is equal to a certain power series $L_W(q)$, defined by specializing one variable in the generating function recording the lengths and absolute lengths of the involutions in W. The simplest inductive method of proving this result for finite Coxeter groups suggests a natural bivariate generalization $L^J_W(s, q) \in \mathbb{Z}[[s, q]]$ depending on a subset $J \subset S$. This new power series specializes to $L_W(q)$ when $s = -1$ and is given explicitly by a sum of rational functions over the involutions which are minimal length representatives of the double cosets of the parabolic subgroup W_J in W. When W is an affine Weyl group, we consider the renormalized power series $T_W(s, q) = L^J_W(s, q)/L_W(q)$ with J given by the generating set of the corresponding finite Weyl group. We show that when W is an affine Weyl group of type A, the power series $T_W(s, q)$ is actually a polynomial in s and q with nonnegative coefficients, which turns out to be a q-analogue recently studied by Cigler of the Chebyshev polynomials of the first kind, arising in a completely different context.

* Corresponding author.
E-mail addresses: emarberg@stanford.edu (E. Marberg), grwhite@math.stanford.edu (G. White).

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1. Introduction

1.1. Background and motivation

Let (W, S) be a Coxeter system with length function $\ell : W \to \mathbb{N}$. The *Poincaré series* of (W, S) is the formal power series (in an indeterminate q) given by

$$P_W(q) = \sum_{w \in W} q^{\ell(w)} \in \mathbb{Z}[[q]].$$

This power series is well-defined if and only if the rank of (W, S) is finite, and in this work, therefore, we require all Coxeter systems (W, S) to have $|S| < \infty$. If W is finite then $P_W(q)$ is obviously a polynomial, and in general $P_W(q)$ is always a rational power series; see [15,28].

Lusztig [21,22] has introduced an interesting analogue of the Poincaré series defined in terms of the twisted involutions in a Coxeter group, and our main object of study here is a natural bivariate generalization of this power series. To motivate its definition, we review some relevant information from [21,22].

To begin, let $\mathrm{Aut}(W, S)$ denote the group of automorphisms of W preserving S, and fix an involution (that is, a self-inverse automorphism) $* \in \mathrm{Aut}(W, S)$. We denote the action of $*$ on elements $w \in W$ by w^*, and write

$$\mathbf{I}_* = \mathbf{I}_*(W) \overset{\text{def}}{=} \{w \in W : w^{-1} = w^*\}$$

for the corresponding set of *twisted involutions* in W. The "twisted" analogue of $P_W(q)$ is the formal power series

$$L_{W,*}(q) = \sum_{w \in \mathbf{I}_*} q^{\ell(w)} \left(\tfrac{q-1}{q+1}\right)^{\ell^*(w)} \in \mathbb{Z}[[q]] \tag{1.1}$$

where on the right side ℓ^* denotes the *twisted absolute length function* defined by Hultman in [14], which is characterized explicitly as the unique map $\mathbf{I}_* \to \mathbb{N}$ such that

(a) $\ell^*(1) = 0$;
(b) ℓ^* is constant on $*$-twisted conjugacy classes, i.e., $\ell^*(sws^*) = \ell^*(w)$ for all $s \in S$;
(c) $\ell^*(ws) - \ell^*(w) = \ell(ws) - \ell(w)$ whenever $s \in S$ and $w \in \mathbf{I}_*$ are such that $ws \in \mathbf{I}_*$.

Note in (c) that $ws \in \mathbf{I}_*$ if and only if $ws = s^*w$. The function ℓ^* is the same as the map denoted ϕ in [21,22]. Lusztig's paper [21, §5.8] appears to be the first place in the literature where the power series (1.1) is considered, and for this reason we denote it by the letter L.