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A double-sum Kronecker-type identity

Eric T. Mortenson [a,b,*]

[a] *Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany*
[b] *Universität zu Köln, Mathematisches Institut, Weyertal 86-90, 50931 Köln, Germany*

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ABSTRACT

We prove a double-sum analog of an identity known to Kronecker and then express it in terms of functions studied by Appell and Kronecker's student Lerch, in so doing we show that the double-sum analog is of mixed mock modular form. We also give related symmetric generalizations.

0. Notation

Let q be a nonzero complex number with $|q| < 1$ and define $\mathbb{C}^* := \mathbb{C} - \{0\}$. Recall

$$(x)_n = (x;q)_n := \prod_{i=0}^{n-1}(1-q^ix), \quad (x)_\infty = (x;q)_\infty := \prod_{i\geq 0}(1-q^ix),$$

* Correspondence to: Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany.
E-mail address: etmortenson@gmail.com.

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$$\text{and} \quad j(x;q) := (x)_\infty (q/x)_\infty (q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n,$$

where in the last line the equivalence of product and sum follows from Jacobi's triple product identity. Here a and m are integers with m positive. Define

$$J_{a,m} := j(q^a;q^m), \quad J_m := J_{m,3m} = \prod_{i\geq 1}(1-q^{mi}), \text{ and } \overline{J}_{a,m} := j(-q^a;q^m).$$

We will use the following definition of an Appell–Lerch function [1,4,8,14]:

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r}{2}} z^r}{1-q^{r-1}xz}. \tag{0.1}$$

1. Introduction

The following identity was known to Kronecker [6], [7, pp. 309–318], see also A. Weil's monograph for Kronecker's proof [12, pp. 70–71]; however, Kronecker's identity is also a special case of Ramanujan's ${}_1\psi_1$-summation. For $x, y \in \mathbb{C}^*$ where $|q| < |x| < 1$ and y neither zero or an integral power of q

$$\sum_{r\in\mathbb{Z}} \frac{x^r}{1-yq^r} = \frac{(q)_\infty^2 (xy, q/xy;q)_\infty}{(x,q/x,y,q/y;q)_\infty}. \tag{1.1}$$

If we place the additional restriction $|q| < |y| < 1$, we have a more symmetric form,

$$\Big(\sum_{r,s\geq 0} - \sum_{r,s<0}\Big) q^{rs}x^r y^s = \frac{J_1^3 j(xy;q)}{j(x;q)j(y;q)}. \tag{1.2}$$

A natural question is what are the higher-dimensional generalizations of (1.1)?

In [4], we expanded Hecke-type double sums in terms of Appell–Lerch functions and theta functions. As an example, we showed for generic $x, y \in \mathbb{C}^*$,

$$\Big(\sum_{r,s\geq 0} - \sum_{r,s<0}\Big)(-1)^{r+s}x^r y^s q^{\binom{r}{2}+2rs+\binom{s}{2}} \tag{1.3}$$
$$= j(y;q)m\big(\frac{q^2x}{y^2},q^3,-1\big) + j(x;q)m\big(\frac{q^2y}{x^2},q^3,-1\big) - \frac{yJ_3^3 j(-x/y;q)j(q^2xy;q^3)}{\overline{J}_{0,3}j(-qy^2/x;q^3)j(-qx^2/y;q^3)}.$$

In [9], we demonstrated how identity (1.2) can by used to determine directly the theta-quotient term of Hecke-type doubles such as in (1.3). Indeed, one can actually see the right-hand side of (1.2) within the extreme right-hand side of (1.3). In trying to determine the modularity of so-called Hecke-type triple-sums [5], i.e. sums of the form