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Perfect necklaces

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ABSTRACT

We introduce a variant of de Bruijn words that we call perfect necklaces. Fix a finite alphabet. Recall that a word is a finite sequence of symbols in the alphabet and a circular word, or necklace, is the equivalence class of a word under rotations. For positive integers k and n, we call a necklace (k, n)-perfect if each word of length k occurs exactly n times at positions which are different modulo n for any convention on the starting point. We call a necklace perfect if it is (k, k)-perfect for some k. We prove that every arithmetic sequence with difference coprime with the alphabet size induces a perfect necklace. In particular, the concatenation of all words of the same length in lexicographic order yields a perfect necklace. For each k and n, we give a closed formula for the number of (k, n)-perfect necklaces. Finally, we prove that every infinite periodic sequence whose period coincides with some (k, n)-perfect necklace for some k and some n, passes all statistical tests of size up to k, but not all larger tests. This last theorem motivated this work.

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1. Introduction

Fix a finite alphabet $\mathcal{A}$ and write $|\mathcal{A}|$ for its cardinality. A word is a finite sequence of symbols in the alphabet. A rotation is the operation that moves the final symbol of a word to the first position while shifting all other symbols to the next position, or it is the composition of this operation with itself an arbitrary number of times. A circular word, or necklace, is the equivalence class of a word under rotations. In this note we introduce *perfect necklaces*.

Definition 1. A necklace is (k, n)-*perfect* if it has length $n|\mathcal{A}|^k$ and each word of length k occurs exactly n times at positions which are different modulo n for any convention on the starting point. A necklace is *perfect* if it is (k, k)-perfect for some k.

Perfect necklaces are a variant of the celebrated de Bruijn necklaces [10]. Recall that a de Bruijn necklace of order k in alphabet $\mathcal{A}$ has length $|\mathcal{A}|^k$ and each word of length k occurs in it exactly once. Thus, our $(k, 1)$-perfect necklaces coincide with the de Bruijn necklaces of order k. For a supreme presentation of de Bruijn necklaces, including a historic account of their discovery and rediscovery, see [5]. Observe that a necklace of length $k|\mathcal{A}|^k$ admits k possible decompositions into $|\mathcal{A}|^k$ consecutive (non-overlapping) words of length k. Hence, a necklace is (k, k)-perfect if and only if it has length $k|A|^k$ and each word of length k occurs exactly once in each of the k possible decompositions.

For each k and n, we give a characterization of (k, n)-perfect necklaces in terms of Eulerian circuits in appropriate graphs (Corollary 14). We give a closed formula for the number of (k, n)-perfect necklaces (Theorem 20). These are the most elaborate results in this work.

We show that each arithmetic sequence with difference coprime with the alphabet size induces a perfect necklace (Theorem 5). In particular, the concatenation of all words of the same length in lexicographic order yields a perfect necklace (Corollary 6). This provides a gracious instance of a perfect necklace for any word length.

The combinatorial properties of the concatenation of all words of the same length in lexicographic order were, as far as we know, considered first by É. Barbier [3,2] (see also [1]). Later Champernowne [8] considered them in his construction of a real number normal to base 10, a property defined by Émile Borel [6]. Champernowne worked with alphabet $\mathcal{A} = \{0, 1, \ldots, 9\}$ and for each k, he bounded the number of occurrences of each word of length up to k in the concatenation of all words of length k in lexicographic order. But neither Barbier nor Champernowne mentioned that each word of length k occurs in this sequence exactly k times, once in each of the k different shifts.

2. Perfect necklaces

Notation We write $\mathcal{A}^*$ for the set of all words, and $\mathcal{A}^k$ for the set of all words of length k. The length of a word w is denoted with $|w|$ and the positions in w are numbered from 0

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