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# Congruences for the number of spin characters of the double covers of the symmetric and alternating groups

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Let $p$ be an odd prime. The bar partitions with sign and $p$-bar-core partitions with sign respectively label the spin characters and $p$-defect zero spin characters of the double cover of the symmetric group, and by restriction, those of the alternating group. The generating functions for these objects have been determined by J. Olsson. We study these functions from an arithmetic perspective, using classical analytic tools and elementary generating function manipulation to obtain many Ramanujan-like congruences.



## 1. Introduction

A partition of $n$ is a non-increasing sequence of positive integers that sum to $n$. Euler determined the generating function for $p(n)$, the number of partitions of $n$:

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$$P(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

The partitions of $n$ label the irreducible characters of the symmetric group $S(n)$. The characters of the alternating group $A(n)$ are then labeled by restriction from $S(n)$. In particular, $p(n)$ gives the dimension of the character table of $S(n)$.

Srinivasa Ramanujan was the first to notice several remarkable arithmetic properties of the partition function.

**Theorem 1.1.** *For all* $n \geq 0$,

$$p(5n+4) = 0 \pmod 5,$$
$$p(7n+5) = 0 \pmod 7,$$
$$p(11n+6) = 0 \pmod{11}.$$

A $t$-core partition of $n$ is a partition in which no hook of size $t$ appears in its Young diagram. When $t$ is prime, the $t$-cores label the $t$-defect zero blocks of $S_n$. In 1990, F. Garvan, D. Kim, and D. Stanton [7] provided a beautiful proof of Theorem 1.1 using core partitions. The generating function for $t$-core partitions has itself become a rich topic of study. For example, work by F. Garvan [6], M. Hirschhorn and J. Sellers [8,9], L. Kolitsch and J. Sellers [13] and N. Baruah and B. Berndt [2], has produced a multitude of Ramanujan-type congruences for $t$-core partitions.

We now consider partitions with distinct parts. For any such partition $\lambda$, shift the $i$th row of its Young diagram $i$ positions to the right. In this shifted diagram $S(\lambda)$ we associate a bar with a certain bar-length to each position. Then we can consider diagrams in which no bar of length $t$ occurs. Partitions of this type will be called $t$-bar-core (or $\bar{t}$-core) partitions. A. O. Morris [15] proved that for $\hat{S}(n)$, the double cover of $S(n)$, bar partitions with sign label the spin characters of $\hat{S}(n)$ and, when $p$ odd, that $\bar{p}$-core partitions with sign label the $p$-defect zero spin characters of $\hat{S}(n)$. Almost three decades later J. Humpreys [10] and M. Cabanes [4], independently, proved a conjecture of Morris on the $p$-block distribution of characters of $\hat{S}(n)$. Spin characters and $p$-defect zero spin characters of $\hat{A}(n)$, the double cover of $A(n)$, are then obtained by restriction from $\hat{S}(n)$, using an application of Clifford theory.

Recent analysis of spin characters of $\hat{S}(n)$ and $\hat{A}(n)$ has focused on their dimensionality (see [3,12]). Here however we study congruences of generating functions for spin characters, $p$-defect zero spin characters of $\hat{S}(n)$ and $\hat{A}(n)$, and $\bar{t}$-cores. The functions for these are known, due to the work of J. Olsson.

In section 2, we introduce the basics: partitions and bar-partitions, bar-core partitions, character theory, generating functions, and some identities. In section 3, we find a characterization of the number of spin characters of $\hat{S}(n)$ and $\hat{A}(n)$ modulo 2 and 3. We then use this to obtain infinitely many Ramanujan-like congruences modulo 2 and 3.