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Major index distribution over permutation classes ☆

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ABSTRACT

For a permutation π the major index of π is the sum of all indices i such that $\pi_i > \pi_{i+1}$. It is well known that the major index is equidistributed with the number of inversions over all permutations of length n. In this paper, we study the distribution of the major index over pattern-avoiding permutations of length n. We focus on the number $M_n^m(\Pi)$ of permutations of length n with major index m, avoiding the set of patterns Π.

First we are able to show that for a singleton set $\Pi = \{\sigma\}$ other than some trivial cases, the values $M_n^m(\Pi)$ are monotonic in the sense that $M_n^m(\Pi) \le M_{n+1}^m(\Pi)$. Our main result is a study of the asymptotic behaviour of $M_n^m(\Pi)$ as n goes to infinity. We prove that for every fixed m, Π and n large enough, $M_n^m(\Pi)$ is equal to a polynomial in n and moreover, we are able to determine the degrees of these polynomials for many sets of patterns.

1. Introduction

Let $\mathcal{S}_n$ be the set of permutations of the letters $\{1, 2, \dots, n\} = [n]$. We write a permutation $\pi \in \mathcal{S}_n$ as a sequence $\pi_1 \cdots \pi_n$. A *permutation statistic* is a function $st : \mathcal{S}_n \to \mathbb{N}_0$.

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For a permutation π, an *inversion* is a pair of different indices $i < j$ such that $\pi_i > \pi_j$ and the *number of inversions* is denoted by $inv(\pi)$. The number of inversions is the oldest and best-known permutation statistic. Already in 1838, Stern [18] proposed a problem of how many inversions there are in all the permutations of length n. The distribution of the number of inversions was given shortly after that by Rodrigues [14].

However, we will focus on a different well-known permutation statistic in this paper. For a permutation π, we say that there is a *descent* on the i-th position if $\pi_i > \pi_{i+1}$. The *major index* of π, denoted by $maj(\pi)$, is then the sum of the positions where the descents occur. The major index statistic is younger than the number of inversions, as it was first defined by MacMahon [12] in 1915. Among other results, MacMahon proved its equidistribution with the number of inversions by showing that their generating functions are equal and started the systematic study of permutation statistics in general. That is why we call the statistics equidistributed with the number of inversions Mahonian. Then it took a long time before Foata [10] proved the equidistribution by constructing his famous bijection. Since then many new Mahonian statistics appeared in the literature, most of which are included in the classification given by Babson and Steingrímsson [1]. For the actual values of Mahonian statistics' distribution see the Mahonian numbers sequence A008302 [15].

We say that two sequences $a_1 \cdots a_n$ and $b_1 \cdots b_n$ are *order-isomorphic* if the permutations required to sort them are the same. A permutation π *contains* a pattern σ if there is a subsequence of $\pi_1 \cdots \pi_n$ order-isomorphic to σ. Otherwise we say that π *avoids* the pattern σ. Pattern avoidance is an active area of research in combinatorics and although the systematic study of pattern avoidance started relatively recently, there is already an extensive amount of literature. A good illustration of an application of pattern avoidance in computer science is that stack-sortable permutations are exactly the ones avoiding pattern 231, which was proved by Knuth [11].

Let $\mathcal{S}_n(\sigma)$ be the set of permutations of length n avoiding σ and $S_n(\sigma)$ its cardinality. We say that patterns σ and τ are *Wilf-equivalent* if $S_n(\sigma) = S_n(\tau)$ for every n. For a permutation statistic st, we say that patterns σ and τ are *st-Wilf-equivalent* if there is a bijection between $\mathcal{S}_n(\sigma)$ and $\mathcal{S}_n(\tau)$ which preserves the statistic st. This refinement of Wilf equivalence has been extensively studied for short patterns of length 3, see [3,4,8,13]. A nearly exhaustive classification of Wilf-equivalence and permutation statistics for patterns of length 3 was given by Claesson and Kitaev [6]. On the other hand, not much is known about permutation statistics and patterns of length 4 and greater. Recently, Dokos et al. [7] presented an in-depth study of major index and number of inversions including st-Wilf-equivalence. They conjectured maj-Wilf-equivalence between certain patterns of length 4, which was proved by Bloom [2]. Another conjecture from Dokos et al. concerning maj-Wilf-equivalent patterns of a specific form was partly proved by Ge, Yan and Zhang [19].

Claesson, Jelínek and Steingrímsson [5] analysed the inversion number distribution over pattern-avoiding classes. Let $I_n^k(\sigma)$ be the number of σ-avoiding permutations with length n and k inversions. Claesson et al. studied $I_n^k(\sigma)$ for a fixed k and a single pattern σ