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Flat matrix models for quantum permutation groups

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ABSTRACT

We study the matrix models $\pi : C(S_N^+) \to M_N(C(X))$ which are flat, in the sense that the standard generators of $C(S_N^+)$ are mapped to rank 1 projections. Our first result is a generalization of the Pauli matrix construction at $N = 4$, using finite groups and 2-cocycles. Our second result is the construction of a universal representation of $C(S_N^+)$, inspired from the Sinkhorn algorithm, that we conjecture to be inner faithful.

Introduction

The quantum permutation group S_N^+ was introduced by Wang in [18]. Of particular interest are the quantum subgroups $\mathcal{G} \subset S_N^+$ appearing from random matrix representations $\pi : C(S_N^+) \to M_N(C(X))$ via the Hopf image construction [2]. One key problem is the computation of the law of the main character of $\mathcal{G}$. See [3,5,6].

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A number of general algebraic and analytic tools for dealing with such questions have been developed [2,6,7,9,19]. However, at the level of concrete examples, only two types of models $\pi : C(S_N^+) \to M_N(C(X))$ have been successfully investigated, so far. The first example, coming from the Pauli matrices, was investigated in [5]. The second example, coming from deformed Fourier matrices, was investigated in [3].

Our purpose here is to advance on such questions:

(1) The Pauli matrix construction and the deformed Fourier matrix one are both of type $\pi : C(S_N^+) \to C(U_B, \mathcal{L}(B))$, with B being a finite dimensional C^*-algebra. We will investigate here the case where $B = C^*_\sigma(G)$ is a cocycle twist of a finite group algebra, which generalizes the Pauli matrix construction. Our main result will be the computation of the law of the main character.
(2) We will present as well a "universal" construction, inspired from the Sinkhorn algorithm [15,16]. This algorithm starts with a $N \times N$ matrix having positive entries and produces, via successive averagings over rows/columns, a bistochastic matrix. We will find here an adaptation of this algorithm to Wang's magic unitaries [18], which conjecturally produces an inner faithful representation of $C(S_N^+)$.

There are of course many questions raised by the present work. Regarding the generalized Pauli matrix construction, our results, and also [1,4], suggest that the associated quantum group should be a twist of PU_n. Also, this construction still remains to be unified with the deformed Fourier matrix one. Regarding the Sinkhorn type models, here our computer simulations suggest that we should get a free Poisson law [13,17], but so far, we have no convincing abstract methods in order to approach this question.

The paper is organized as follows: 1–2 contain preliminaries and generalities, in 3–4 we study the generalized Pauli models, and in 5–6 we study the Sinkhorn type models.

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1. Quantum permutations

We are interested in what follows in the quantum permutation group S_N^+, and in the random matrix representations of the associated Hopf algebra $C(S_N^+)$.

Our starting point is the following notion, coming from Wang's paper [18]:

Definition 1.1. A magic unitary is a square matrix over a C^*-algebra, $u \in M_N(A)$, whose entries are projections, summing up to 1 on each row and each column.