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$*$-freeness in finite tensor products

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In this paper, we consider the following question and variants thereof: given $\mathbf{D} := \big(a_{1;i} \otimes \cdots \otimes a_{K;i} : i \in I\big)$, a collection of elementary tensor non-commutative random variables in the tensor product of probability spaces $(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_K, \varphi_1 \otimes \cdots \otimes \varphi_K)$, when is $\mathbf{D}$ $*$-free? (See Section 1.2 for a precise formulation of this problem.)
Settling whether or not freeness occurs in tensor products is a recurring problem in operator algebras, and the following two examples provide a natural motivation for the above question:

(A) If $(a_{1;i} : i \in I)$ is a $*$-free family of Haar unitary variables and $a_{k,i}$ are arbitrary unitary variables for $k \geq 2$, then the $*$-freeness persists at the level of the tensor product $\mathbf{D}$.
(B) A converse of (A) holds true if all variables $a_{k;i}$ are group-like elements (see Corollary 1.7 of Proposition 1.6).

It is therefore natural to seek to understand the extent to which such simple characterizations hold true in more general cases. While our results fall short of a complete characterization, we make notable steps toward identifying necessary and sufficient conditions for the freeness of $\mathbf{D}$. For example, we show that under evident assumptions, if more

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than one family $(a_{k,i} : i \in I)$ contains non-unitary variables, then the tensor family fails to be $*$-free (see Theorem 1.8 (1)).

1. Introduction

1.1. Motivating observations

In connection with recent investigations on quantum expanders and related topics in operator algebras [6–8], several years ago G. Pisier and R. Speicher asked the following question to the first named author: given $U_1^{(N)}, \ldots, U_n^{(N)}$, n independent Haar distributed $N \times N$ unitary random matrices, are

$$U_1^{(N)} \otimes \overline{U_1^{(N)}}, \ldots, U_n^{(N)} \otimes \overline{U_n^{(N)}}$$

asymptotically $*$-free as $N \to \infty$ ($\overline{U_i^{(N)}}$ denotes the entrywise complex conjugate)?

Thanks to the almost sure asymptotic $*$-freeness of $(U_1^{(N)}, \ldots, U_n^{(N)})$ (see [2], for instance), a simple argument shows that the above question can be answered in the affirmative: for any collection $(V_1^{(N)}, \ldots, V_n^{(N)})$ that converges in joint $*$-distribution to a collection $(v_1, \ldots, v_n)$ of (not necessarily $*$-free) unitary variables in an arbitrary non-commutative $*$-probability space, $(U_1^{(N)} \otimes V_1^{(N)}, \ldots, U_n^{(N)} \otimes V_n^{(N)})$ is almost surely asymptotically $*$-free. This follows directly from the definition of asymptotic $*$-freeness and the fact that $(U_1^{(N)}, \ldots, U_n^{(N)})$ converges to a collection of $*$-free Haar unitaries (see Proposition 1.5 and Section 3 for a detailed proof of a more general version of this result). Then, taking $V_i^{(N)} = \overline{U_i^{(N)}}$ solves the above question, and a version in expectation can be achieved thanks to the Dominated Convergence Theorem.

Remark 1.1. Note that the above reasoning cannot be used to similarly extend results of strong asymptotic freeness of random matrices (such as [1,4]) to the strong asymptotic freeness of tensor products of random matrices. Characterizing the occurrence of strong convergence in tensor products remains an unsolved and seemingly difficult problem.

While taking the $V_i^{(N)}$ to be unitary is natural given the present applications in operator algebras, one may wonder if a similar phenomenon occurs when the unitarity assumption is dropped. Understanding the mechanisms that give rise to $*$-freeness in general tensor products turns out to be a very interesting and surprisingly difficult question with connections to group theory, which is what we explore in this paper.

1.2. Main problem

For a fixed $K \in \mathbb{N}$, let $(\mathcal{A}_1, \varphi_1), \ldots, (\mathcal{A}_K, \varphi_K)$ be $*$-probability spaces, and let $(\mathcal{A}, \varphi) = (\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_K, \varphi_1 \otimes \cdots \otimes \varphi_K)$ be their tensor product (in which the $\mathcal{A}_k$ are independent in