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Reconstruction of n-dimensional convex bodies from surface tensors

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ABSTRACT

In this paper, we derive uniqueness and stability results for surface tensors. Further, we develop two algorithms that reconstruct shape of n-dimensional convex bodies. One algorithm requires knowledge of a finite number of surface tensors, whereas the other algorithm is based on noisy measurements of a finite number of harmonic intrinsic volumes. The derived stability results ensure consistency of the two algorithms. Examples that illustrate the feasibility of the algorithms are presented.

1. Introduction

Recently, Minkowski tensors have successfully been used as shape descriptors of spatial structures in materials science, see, e.g., [3,15,16]. Surface tensors are translation invariant Minkowski tensors derived from surface area measures, and the shape of a con-

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vex body K with nonempty interior in $\mathbb{R}^n$ is uniquely determined by the surface tensors of K. In this context, the shape of K is defined as the equivalence class of all translations of K.

In [10], Kousholt and Kiderlen develop reconstruction algorithms that approximate the shape of convex bodies in $\mathbb{R}^2$ from a finite number of surface tensors. Two algorithms are described. One algorithm requires knowledge of exact surface tensors, and one allows for noisy measurements of surface tensors. For the latter algorithm, it is argued that it is preferable to use harmonic intrinsic volumes instead of surface tensors evaluated at the standard basis. The similar problem of reconstructing a convex body K from its volume tensors (moments of the Lebesgue measure restricted to K) has received considerable attention and can be applied in X-ray tomography, see, e.g., [12,8].

The purpose of this paper is threefold. Firstly, the reconstruction algorithms developed in [10] are generalized to an n-dimensional setting. Secondly, stability and uniqueness results for surface tensors are established, and the stability results are used to ensure consistency of the generalized algorithms. Thirdly, we illustrate the feasibility of the reconstruction algorithms by examples. The generalizations of the reconstruction algorithms are developed along the same lines as the algorithms for convex bodies in $\mathbb{R}^2$. However, there are several non-trivial obstacles on the way. In particular, essentially different stability results are needed to ensure consistency.

The input of the first generalized algorithm is exact surface tensors up to a certain rank of an unknown convex body in $\mathbb{R}^n$. The output is a polytope with surface tensors identical to the given surface tensors of the unknown convex body. The input of the second generalized algorithm is (possibly noisy) measurements of harmonic intrinsic volumes of an unknown convex body in $\mathbb{R}^n$, and the output is a polytope with harmonic intrinsic volumes that fit the given measurements in a least squares sense. When $n \geq 3$, a convex body that fits the noisy input measurements of harmonic intrinsic volumes may not exist, and in this case, the output of the algorithm based on harmonic intrinsic volumes is a message stating that there is no solution to the given task. However, this situation only occurs when the measurements are too noisy, see Lemma 6.3.

The consistency of the algorithms described in [10] is established using the stability result [10, Thm. 4.8] for harmonic intrinsic volumes derived from the first order area measure. This result can be applied as the first order area measure and the surface area measure coincide for $n = 2$. However, for $n \geq 3$, the stability result is not applicable. Therefore, we establish stability results for surface tensors and for harmonic intrinsic volumes derived from surface area measures. More precisely, first we derive an upper bound of the Dudley distance between surface area measures of two convex bodies. This bound is small, when the distance between the harmonic intrinsic volumes up to degree s of the convex bodies is small for some large $s \in \mathbb{N}_0$ (Theorem 4.3). From this result and a known connection between the Dudley distance and the translative Hausdorff distance, we obtain that the translative Hausdorff distance between convex bodies with identical surface tensors up to rank s becomes small, when s is large (Corollary 4.4). The structures of the two stability result differ. The first result allows the difference between