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Integral geometry of translation invariant functionals, II: The case of general convex bodies

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A R T I C L E I N F O

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A B S T R A C T

In continuation of Part I, we study translative integral formulas for certain translation invariant functionals, which are defined on general convex bodies. Again, we consider local extensions and use these to show that the translative formulas extend to arbitrary continuous and translation invariant valuations. Then, we discuss applications to Poisson particle processes and Boolean models which contain, as a special case, some new results for flag measures.

1. Introduction

In the introduction to the first part of this paper [20], we have motivated the study of translation invariant functionals φ on $\mathcal{K}$, the class of convex bodies in $\mathbb{R}^d$, which have a local extension as a kernel. For functionals on $\mathcal{P}$, the set of polytopes, these local functionals provide a richer class than the classical additive functionals (*valuations*). A major reason for studying local functionals was the fact that they obey translative integral formulas similar to the integral geometric results for curvature measures and

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intrinsic volumes. We have seen that there are some interesting new functionals contained in this local concept, for example the total k-volume of the k-skeleton of a polytope, $k \in \{0, ..., d-1\}$. Although the local functionals on $\mathcal{P}$ need not be additive, they allow an extension to unions of polytopes which are in mutual general position. This was used in [20] to obtain formulas for local functionals of Boolean models which are in analogy to the well-known results for intrinsic volumes (as they are presented in [16, Section 9.1]).

In this second part, we discuss local functionals φ on $\mathcal{K}$. It is natural then to add a continuity condition (in order to use polytopal approximation, for example). As we shall see, this changes the situation drastically. Namely, a (continuous) local functional on $\mathcal{K}$ is automatically additive, hence a valuation. In the opposite direction, for any translation invariant continuous valuation φ on $\mathcal{K}$, the restriction to $\mathcal{P}$ is a local functional. However, this does not guarantee that φ admits a local extension on $\mathcal{K}$. In fact, the question whether every translation invariant continuous valuation on $\mathcal{K}$ is a local functional remains open.

We shall show, that there is again a translative integral formula for local functionals on $\mathcal{K}$ which can be iterated. This even holds for valuations (without the assumption on a local extension). Thus, for a translation invariant, continuous and j-homogeneous valuation $\varphi^{(j)}$ on $\mathcal{K}$, we obtain a sequence of mixed functionals $\varphi^{(j)}_{m_1,\dots,m_k}$ which are in analogy to the mixed functionals $V^{(j)}_{m_1,\dots,m_k}$ arising from the intrinsic volumes V_j, $j = 0, ..., d-1$. We then study kinematic formulas and we extend some of the results from the first part to mean values of valuations for Boolean models with general convex or polyconvex grains. Finally, we discuss a few special cases and obtain new integral and mean value formulas for flag measures. In an appendix we provide an approximation result (with a proof by Rolf Schneider) which is used in Section 3.

2. Definitions and basic result

For completeness, we repeat some of the notations and definitions used in [20]. For general notions from convex geometry, we refer to [15].

Let $\mathcal{K}$ be the space of convex bodies in $\mathbb{R}^d$ supplied with the Hausdorff metric, let $\mathcal{P}$ be the (dense) subset of convex polytopes and let $\mathcal{B}$ denote the σ-algebra of Borel sets in $\mathbb{R}^d$. For a convex body K and $j = 0, ..., d$, $V_j(K)$ denotes the jth intrinsic volume and $\Phi_j(K, \cdot)$ is the jth curvature measure. Thus, $V_d(K) = \lambda(K)$ is the volume of K (and λ is the Lebesgue measure in $\mathbb{R}^d$). We denote by λ_K the restriction of λ to K. For a polytope P, let $\mathcal{F}_j(P)$ be the collection of j-faces of P, $j = 0, ..., d-1$, and let $n(P, F)$, for a face $F \in \mathcal{F}_j(P)$, be the intersection of the normal cone $N(P, F)$ of P at F with the unit sphere S^{d-1}; this is a member of $\wp^{d-1}_{d-j-1}$, the class of $(d-j-1)$-dimensional spherical polytopes. Later, we will also use the larger class $\tilde{\wp}^{d-1}_{d-j-1}$, which consists of the spherical polytopes of dimension $\le d-j-1$. For $F \in \mathcal{F}_j(P)$, let λ_F be the restriction to F of the (j-dimensional) Lebesgue measure in the affine hull of F. Here, the dimension of λ_F will always be clear from the context.