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# Connectivity functions and polymatroids

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ABSTRACT

A *connectivity function on* a set $E$ is a function $\lambda : 2^E \to \mathbb{R}$ such that $\lambda(\emptyset) = 0$, that $\lambda(X) = \lambda(E - X)$ for all $X \subseteq E$ and that $\lambda(X \cap Y) + \lambda(X \cup Y) \leq \lambda(X) + \lambda(Y)$ for all $X, Y \subseteq E$. Graphs, matroids and, more generally, polymatroids have associated connectivity functions. We introduce a notion of duality for polymatroids and prove that every connectivity function is the connectivity function of a self-dual polymatroid. We also prove that every integral connectivity function is the connectivity function of a half-integral self-dual polymatroid.



## 1. Introduction

Let $E$ be a finite set, and $\lambda$ be a function from the power set of $E$ into the real numbers. Then $\lambda$ is *symmetric* if $\lambda(X) = \lambda(E - X)$ for all $X \subseteq E$; $\lambda$ is *submodular* if $\lambda(X \cap Y) + \lambda(X \cup Y) \leq \lambda(X) + \lambda(Y)$ for all $X, Y \subseteq E$; and $\lambda$ is *normalised* if $\lambda(\emptyset) = 0$. If $\lambda$ is symmetric, submodular and normalised, then we say that $\lambda$ is a *connectivity*

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*function* with *ground set* $E$. We also say that $\lambda$ is a connectivity function *on* $E$. If $\lambda$ is a connectivity function on $E$, then $\lambda$ is *integer-valued* if $\lambda(X) \in \mathbb{Z}$ for all $X \subseteq E$. The connectivity function $\lambda$ is *unitary* if $\lambda(\{x\}) \leq 1$ for all $x \in E$.

Graphs and matroids have natural associated connectivity functions. These auxiliary structures capture vital information. It turns out that a number of quite fundamental properties of graphs and matroids hold at the level of general connectivity functions. In particular this is the case for properties associated with branch-width and tangles of graphs and matroids. This is implicit — but clear on a close reading — in the paper of Robertson and Seymour [11]. More explicit results for connectivity functions are proved in Geelen, Gerards and Whittle [2], Clark and Whittle [1], Hundertmark [5], and Grohe and Schweitzer [3].

Given that we can prove quite strong theorems for connectivity functions, the study of these structures is well motivated and this paper forms part of that study. The natural question arises as to just how general connectivity functions are. The main purpose of this paper is to give an answer to that question. Polymatroids are defined in the next section. We prove that every connectivity function is the connectivity function of an associated polymatroid, and every integral connectivity function is the connectivity function of an associated half-integral polymatroid. The proofs of these facts are quite simple — almost unnervingly so — but the results are apparently new and we believe that they are worth reporting. Moreover, our main result surprised at least one of us as a number of naturally arising connectivity functions seem to have little to do with polymatroids.

As well as proving the above results we introduce a new notion of duality for polymatroids. Via this duality we get stronger theorems. Every connectivity function is the connectivity function of a *self-dual* polymatroid. An interesting feature of this notion of duality is that, when restricted to the class of matroids, it gives a duality that is subtly different from usual matroid duality.

The results of this paper had their genesis in the MSc thesis of Mo [9] and were further developed in the MSc thesis of Jowett [6]. These theses also contain a number of other results on connectivity functions and their connection with polymatroids.

Since writing the first draft of this paper we have become aware of a paper of Matúš [8]. While our perspective and terminology are quite different from those of Matúš the fact is that a number of the results of this paper follow from results of his. In particular our Lemmas 4.2 and 4.3 (ii) and (iii) follow from Theorem 1 of [8]. At a deeper level it is clear that most of the key ideas for which this paper could claim originality are already present in [8]. The existence of Matúš' paper came as a considerable surprise to us as we believed throughout that we were exploring perfectly new territory. On the other hand Matúš' perspective is quite different from ours — he is motivated by problems in information theory and our primary motivation comes from matroid theory. Moreover the two papers have very different styles of exposition. The two papers should appeal to different audiences and we believe that there is a real advantage in having both papers in print.