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Additive representation of symmetric inverse M-matrices and potentials

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In this article we characterize the closed cones respectively generated by the symmetric inverse M-matrices and by the inverses of symmetric row diagonally dominant M-matrices. We show the latter has a finite number of extremal rays, while the former has infinitely many extremal rays. As a consequence we prove that every potential is the sum of ultrametric matrices.

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1. Introduction

Consider the cone $\mathcal{K}$ generated by the set of symmetric inverse M-matrices and the cone $\mathcal{KP}$ generated by potentials, that is, inverses of symmetric diagonally dominant M-matrices. We study these cones to understand the difference between inverse M-matrices and potentials. To our surprise, $\mathcal{KP}$ has only a finite number of extremal rays, that is, $\mathcal{KP}$ is a polyhedral cone, while $\mathcal{K}$ has infinitely many extremal rays.

The extremal rays of $\mathcal{K}$ are generated by the rank one matrices uu' where u is a nonnegative nonzero vector. This is shown in Theorem 2.1. On the other hand, the extremal rays of $\mathcal{KP}$ are the rank one matrices uu', where u is a $\{0, 1\}$-valued nonzero vector. This is shown in Theorem 3.2. While the first result is simple to show, the second one is more involved and its proof relies on some properties of the adjoint of the symmetric polynomial matrix

$$M(n, (\mathbf{Y}, \mathbf{Z})) = \begin{pmatrix} y_1 + S_1 & -z_{12} & -z_{13} & \cdots & -z_{1n} \\ -z_{21} & y_2 + S_2 & -z_{23} & \cdots & -z_{2n} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ -z_{n-1,1} & -z_{n-1,2} & \cdots & y_{n-1} + S_{n-1} & -z_{n-1,n} \\ -z_{n,1} & -z_{n,2} & \cdots & -z_{n,n-1} & y_n + S_n \end{pmatrix}, \tag{1.1}$$

where: $\mathbf{Y} = (y_1, \cdots, y_n)$, $\mathbf{Z} = (z_{ij} : i, j = 1, \cdots n, i \neq j)$ with $z_{ij} = z_{ji}$ and $S_i = S_i(\mathbf{Z}) = \sum\limits_{j \neq i} z_{ij}$. Sometimes we write $\mathbf{X} = (\mathbf{Y}, \mathbf{Z})$ and $M(n, \mathbf{X}) = M(n, (\mathbf{Y}, \mathbf{Z}))$.

The important property of $V = adj(M)$ is the minimality of it, which simply says that V_{ij}, for $i \neq j$, is the intersection of the two polynomials V_{ii} and V_{jj} (see Definition 3.4). We also use results of Wang's algebra, which simplifies some of our computations.

In Appendix B, we include some historical remarks about the principal minors of $M(n, \mathbf{X})$ and its determinant.

2. Representation for inverse M-matrices

We fix $I = \{1, \cdots, n\}$. An M-matrix is a nonsingular matrix, whose off-diagonal elements are nonpositive and its inverse is a nonnegative matrix (every entry is nonnegative). Given a nonnegative matrix U, an important problem is to characterize (in terms of U) when it is the inverse of an M-matrix. In this direction, in the next result we study the cone generated by inverses of symmetric M-matrices.

Theorem 2.1. *Assume that U is the inverse of a symmetric M-matrix of order n. Then, there exist n nonnegative and linearly independent vectors $v_1, \cdots, v_n \in \mathbb{R}^n$, such that*

$$U = \sum_{k=1}^{n} v_k v_k'. \tag{2.1}$$