

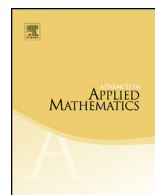


ELSEVIER

Contents lists available at ScienceDirect

Advances in Applied Mathematics

[www.elsevier.com/locate/yaama](http://www.elsevier.com/locate/yaama)



## Additive representation of symmetric inverse $M$ -matrices and potentials



Claude Dellacherie<sup>a</sup>, Servet Martinez<sup>b</sup>, Jaime San Martin<sup>b,\*</sup>

<sup>a</sup> *Laboratoire Raphaël Salem, UMR 6085, Université de Rouen, Site du Madrillet, 76801 Saint Étienne du Rouvray Cedex, France*

<sup>b</sup> *CMM-DIM, Universidad de Chile, UMI-CNRS 2807, Casilla 170-3 Correo 3, Santiago, Chile*

### ARTICLE INFO

#### Article history:

Received 7 April 2015

Received in revised form 11 April 2016

Accepted 21 April 2016

Available online 17 June 2016

#### MSC:

15A48

15A51

60J45

#### Keywords:

$M$ -matrix

Potential matrix

Ultrametric matrices

Wang algebra

Matrix-tree theorem

### ABSTRACT

In this article we characterize the closed cones respectively generated by the symmetric inverse  $M$ -matrices and by the inverses of symmetric row diagonally dominant  $M$ -matrices. We show the latter has a finite number of extremal rays, while the former has infinitely many extremal rays. As a consequence we prove that every potential is the sum of ultrametric matrices.

© 2016 Elsevier Inc. All rights reserved.

\* Corresponding author.

*E-mail addresses:* [Claude.Dellacherie@univ-rouen.fr](mailto:Claude.Dellacherie@univ-rouen.fr) (C. Dellacherie), [smartine@dim.uchile.cl](mailto:smartine@dim.uchile.cl) (S. Martinez), [jsanmart@dim.uchile.cl](mailto:jsanmart@dim.uchile.cl) (J. San Martin).

### 1. Introduction

Consider the cone  $\mathcal{K}$  generated by the set of symmetric inverse  $M$ -matrices and the cone  $\mathcal{KP}$  generated by potentials, that is, inverses of symmetric diagonally dominant  $M$ -matrices. We study these cones to understand the difference between inverse  $M$ -matrices and potentials. To our surprise,  $\mathcal{KP}$  has only a finite number of extremal rays, that is,  $\mathcal{KP}$  is a polyhedral cone, while  $\mathcal{K}$  has infinitely many extremal rays.

The extremal rays of  $\mathcal{K}$  are generated by the rank one matrices  $uu'$  where  $u$  is a nonnegative nonzero vector. This is shown in [Theorem 2.1](#). On the other hand, the extremal rays of  $\mathcal{KP}$  are the rank one matrices  $uu'$ , where  $u$  is a  $\{0, 1\}$ -valued nonzero vector. This is shown in [Theorem 3.2](#). While the first result is simple to show, the second one is more involved and its proof relies on some properties of the adjoint of the symmetric polynomial matrix

$$M(n, (\mathbf{Y}, \mathbf{Z})) = \begin{pmatrix} y_1 + S_1 & -z_{12} & -z_{13} & \cdots & -z_{1n} \\ -z_{21} & y_2 + S_2 & -z_{23} & \cdots & -z_{2n} \\ \vdots & \vdots & \ddots & \dots & \vdots \\ -z_{n-1,1} & -z_{n-1,2} & \cdots & y_{n-1} + S_{n-1} & -z_{n-1,n} \\ -z_{n,1} & -z_{n,2} & \cdots & -z_{n,n-1} & y_n + S_n \end{pmatrix}, \tag{1.1}$$

where:  $\mathbf{Y} = (y_1, \dots, y_n)$ ,  $\mathbf{Z} = (z_{ij} : i, j = 1, \dots, n, i \neq j)$  with  $z_{ij} = z_{ji}$  and  $S_i = S_i(\mathbf{Z}) = \sum_{j \neq i} z_{ij}$ . Sometimes we write  $\mathbf{X} = (\mathbf{Y}, \mathbf{Z})$  and  $M(n, \mathbf{X}) = M(n, (\mathbf{Y}, \mathbf{Z}))$ .

The important property of  $V = adj(M)$  is the minimality of it, which simply says that  $V_{ij}$ , for  $i \neq j$ , is the intersection of the two polynomials  $V_{ii}$  and  $V_{jj}$  (see [Definition 3.4](#)). We also use results of Wang’s algebra, which simplifies some of our computations.

In [Appendix B](#), we include some historical remarks about the principal minors of  $M(n, \mathbf{X})$  and its determinant.

### 2. Representation for inverse $M$ -matrices

We fix  $I = \{1, \dots, n\}$ . An  $M$ -matrix is a nonsingular matrix, whose off-diagonal elements are nonpositive and its inverse is a nonnegative matrix (every entry is nonnegative). Given a nonnegative matrix  $U$ , an important problem is to characterize (in terms of  $U$ ) when it is the inverse of an  $M$ -matrix. In this direction, in the next result we study the cone generated by inverses of symmetric  $M$ -matrices.

**Theorem 2.1.** *Assume that  $U$  is the inverse of a symmetric  $M$ -matrix of order  $n$ . Then, there exist  $n$  nonnegative and linearly independent vectors  $v_1, \dots, v_n \in \mathbb{R}^n$ , such that*

$$U = \sum_{k=1}^n v_k v_k'. \tag{2.1}$$

Download English Version:

<https://daneshyari.com/en/article/4624492>

Download Persian Version:

<https://daneshyari.com/article/4624492>

[Daneshyari.com](https://daneshyari.com)