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Log-convex and Stieltjes moment sequences

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ABSTRACT

We show that Stieltjes moment sequences are infinitely log-convex, which parallels a famous result that (finite) Pólya frequency sequences are infinitely log-concave. We introduce the concept of q-Stieltjes moment sequences of polynomials and show that many well-known polynomials in combinatorics are such sequences. We provide a criterion for linear transformations and convolutions preserving Stieltjes moment sequences. Many well-known combinatorial sequences are shown to be Stieltjes moment sequences in a unified approach and therefore infinitely log-convex, which in particular settles a conjecture of Chen and Xia about the infinite log-convexity of the Schröder numbers. We also list some interesting problems and conjectures about the log-convexity and the Stieltjes moment property of the (generalized) Apéry numbers.

1. Introduction

Let $\alpha = (a_k)_{k\geq 0}$ be a sequence of nonnegative numbers. The sequence is called *log-convex* (*log-concave*, resp.) if $a_k a_{k+2} \geq a_{k+1}^2$ ($a_k a_{k+2} \leq a_{k+1}^2$, resp.) for all $k \geq 0$. The

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log-convex and log-concave sequences arise often in combinatorics and have been extensively investigated. We refer the reader to [7,23,25] for the log-concavity and [17,27] for the log-convexity. A basic approach to such problems comes from the theory of total positivity [6–11].

Let $A = [a_{n,k}]_{n,k\geq 0}$ be a finite or infinite matrix of real numbers. It is called *totally positive* (*TP* for short) if all its minors are nonnegative. It is called TP_2 if all minors of order ≤ 2 are nonnegative. Given a sequence $\alpha = (a_k)_{k\geq 0}$, define its *Toeplitz matrix* $T(\alpha)$ and *Hankel matrix* $H(\alpha)$ by

$$T(\alpha) = [a_{i-j}]_{i,j\geq 0} = \begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & & \cdots & & \ddots \end{bmatrix}$$

and

$$H(\alpha) = [a_{i+j}]_{i,j\geq 0} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \\ a_2 & a_3 & a_4 & a_5 & \\ a_3 & a_4 & a_5 & a_6 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

Clearly, a sequence of positive numbers is log-concave (log-convex, resp.) if and only if the corresponding Toeplitz matrix (Hankel matrix, resp.) is TP_2.

We say that α is a *Pólya frequency sequence* (*PF* for short) if its Toeplitz matrix $T(\alpha)$ is TP. Such sequences have been deeply studied in the theory of total positivity [15] and in combinatorics [6]. For example, the fundamental representation theorem of Schoenberg and Edrei states that a sequence $a_0 = 1, a_1, a_2, \ldots$ of real numbers is PF if and only if its generating function has the form

$$\sum_{k\geq 0} a_k x^k = e^{\gamma z} \frac{\prod_{j\geq 1}(1+\alpha_j z)}{\prod_{j\geq 1}(1-\beta_j z)}$$

in some open disk centered at the origin, where $\alpha_j, \beta_j, \gamma \geq 0$ and $\sum_{j\geq 1}(\alpha_j + \beta_j) < +\infty$ (see [15, p. 412] for instance). In particular, a finite sequence of nonnegative numbers is PF if and only if its generating function has only real zeros [15, p. 399].

We say that $\alpha = (a_k)_{k\geq 0}$ is a *Stieltjes moment* (*SM* for short) sequence if its Hankel matrix $H(\alpha)$ is TP. It is well known that α is a Stieltjes moment sequence if and only if it has the form