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Extreme values of the stationary distribution of random walks on directed graphs

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ABSTRACT

We examine the stationary distribution of random walks on directed graphs. In particular, we focus on the *principal ratio*, which is the ratio of maximum to minimum values of vertices in the stationary distribution. We give an upper bound for this ratio over all strongly connected graphs on n vertices. We characterize all graphs achieving the upper bound and we give explicit constructions for these extremal graphs. Additionally, we show that under certain conditions, the principal ratio is tightly bounded. We also provide counterexamples to show the principal ratio cannot be tightly bounded under weaker conditions.

1. Introduction

In the study of random walks on graphs, many problems that are straightforward for undirected graphs are relatively complicated in the directed case. One basic problem

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concerns determining stationary distributions of random walks on simple directed graphs. For an undirected graph, the vector $\pi(v) = \frac{d_v}{\sum_v d_v}$, where d_v is the degree of vertex v, is the unique stationary distribution if the graph is connected and non-bipartite. Consequently, the principal ratio, which is the ratio of maximum to minimum values of vertices in the stationary distribution, is $\frac{\max_v d_v}{\min_v d_v}$ and thus is at most n, the number of vertices.

In contrast, the directed case is far more subtle: not only does no such closed form solution exist for the stationary distribution, but its principal ratio can be exponentially large in n. This has immediate implications for the central question of bounding the rate of convergence of a random walk on a directed graph where extreme values of the stationary distribution play an important role in addition to eigenvalues. For example, it can be shown that for a strongly connected directed graph, the order of the rate of convergence is bounded above by $2\lambda_1^{-1}(-\log(\min_x \pi(x)))$, where λ_1 is the first nontrivial eigenvalue of the normalized Laplacian of the directed graph, as defined in [3]. Namely after at most $t \geq 2\lambda_1^{-1}(-\log(\min_x \pi(x)) + 2c)$ steps, the total variation distance is at most e^{-c}.

Another application of the stationary distribution and its principal ratio is in the algorithmic design and analysis of vertex ranking, for so-called "*PageRank*" algorithms for directed graphs (since many real-world information networks are indeed directed graphs). PageRank algorithms [2] use a variation of random walks with an additional diffusion parameter and therefore it is not surprising that the effectiveness of the algorithm depends on the principal ratio.

In addition to its role in Page Rank algorithmic analysis and bounding the rate of converge in random walks, it has been noted (see [4]) that the principal ratio can be interpreted as a numerical metric for graph irregularity since it achieves its minimum of 1 only for regular graphs.

The study of the principal ratio of the stationary distribution has a rich history. We note that the stationary distribution is a special case of the Perron vector, which is the unique positive eigenvector associated with the largest eigenvalue of an irreducible matrix with non-negative entries. There is a large literature examining the Perron vector of the adjacency matrix of undirected graphs, which has been studied by Cioabă and Gregory [4], Tait and Tobin [10], Papendieck and Recht [9], Zhao and Hong [12], and Zhang [11]. In this paper, we focus on principal ratio of the stationary distribution of random walk on a strongly connected directed graph with n vertices.

For directed graphs, some relevant prior results are from matrix analysis. Latham [5], Minc [7], and Ostrowski [8] studied the Perron vector of a (not necessarily symmetric) matrix with positive entries, which can be used to study matrices associated with complete, weighted directed graphs. However, for our case a relevant prior result comes from Lynn and Timlake, who gave bounds of the principal ratio for *primitive* matrices with non-negative entries (see Corollary 2.1.1 in [6]). As we will soon further explain, since ergodic random walks on directed graphs have primitive transition probability matrices, their result applies naturally in our setting. Letting $\gamma(D)$ denote the principal ratio of a directed graph D, their result yields the bound