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A new bijection relating q-Eulerian polynomials

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ABSTRACT

On the set of permutations of a finite set, we construct a bijection which maps the 3-vector of statistics $(\mathrm{maj} - \mathrm{exc}, \mathrm{des}, \mathrm{exc})$ to a 3-vector $(\mathrm{maj}_2, \widetilde{\mathrm{des}}_2, \mathrm{inv}_2)$ associated with the q-Eulerian polynomials introduced by Shareshian and Wachs (2014) [10].

0. Notations

For all pair of integers (n, m) such that $n < m$, the set $\{n, n+1, \ldots, m\}$ is indifferently denoted by $[n, m]$, $]n-1, m]$, $[n, m+1[$ or $]n-1, m+1[$.

The set of positive integers $\{1, 2, 3, \ldots\}$ is denoted by $\mathbb{N}_{>0}$.

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For all integer $n \in \mathbb{N}_{>0}$, we denote by $[n]$ the set $[1, n]$ and by $\mathfrak{S}_n$ the set of the permutations of $[n]$. By abuse of notation, we assimilate every $\sigma \in \mathfrak{S}_n$ with the word $\sigma(1)\sigma(2)\dots\sigma(n)$.

If a set $S = \{n_1, n_2, \dots, n_k\}$ of integers is such that $n_1 < n_2 < \dots < n_k$, we sometimes use the notation $S = \{n_1 < n_2 < \dots < n_k\}$.

1. Introduction

Let n be a positive integer and $\sigma \in \mathfrak{S}_n$. A *descent* (resp. *excedance point*) of σ is an integer $i \in [n-1]$ such that $\sigma(i) > \sigma(i+1)$ (resp. $\sigma(i) > i$). The set of descents (resp. excedance points) of σ is denoted by $\mathrm{DES}(\sigma)$ (resp. $\mathrm{EXC}(\sigma)$) and its cardinal by $\mathrm{des}(\sigma)$ (resp. $\mathrm{exc}(\sigma)$). The integers $\sigma(i)$ with $i \in \mathrm{EXC}(\sigma)$ are called excedance values of σ.

It is due to MacMahon [5] and Riordan [7] that

$$\sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)} = A_n(t),$$

where $A_n(t)$ is the n-th Eulerian polynomial [1]. A statistic equidistributed with des or exc is said to be *Eulerian*. The statistic ides defined by $\mathrm{ides}(\sigma) = \mathrm{des}(\sigma^{-1})$ obviously is Eulerian.

The *major index* of a permutation $\sigma \in \mathfrak{S}_n$ is defined as

$$\mathrm{maj}(\sigma) = \sum_{i \in \mathrm{DES}(\sigma)} i.$$

It is also due to MacMahon that

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} = \prod_{i=1}^{n} (1-q^i)/(1-q).$$

A statistic equidistributed with maj is said to be *Mahonian*. Among Mahonian statistics is the statistic inv, defined by $\mathrm{inv}(\sigma) = |\mathrm{INV}(\sigma)|$ where $\mathrm{INV}(\sigma)$ is the set of *inversions* of a permutation $\sigma \in \mathfrak{S}_n$, *i.e.* the pairs of integers $(i, j) \in [n]^2$ such that $i < j$ and $\sigma(i) > \sigma(j)$.

In [10], Shareshian and Wachs consider analogous versions (introduced in [6]) of the above statistics: let $\sigma \in \mathfrak{S}_n$, the set of *2-descents* (resp. *2-inversions*) of σ is defined as

$$\mathrm{DES}_2(\sigma) = \{i \in [n-1] : \sigma(i) > \sigma(i+1) + 1\}$$

(resp.

$$\mathrm{INV}_2(\sigma) = \{1 \leq i < j \leq n : \sigma(i) = \sigma(j) + 1\})$$

and its cardinal is denoted by $\mathrm{des}_2(\sigma)$ (resp. $\mathrm{inv}_2(\sigma)$).