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On the free Lie algebra with multiple brackets ☆

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ABSTRACT

It is a classical result that the multilinear component of the free Lie algebra is isomorphic (as a representation of the symmetric group) to the top (co)homology of the proper part of the poset of partitions Π_n tensored with the sign representation. We generalize this result in order to study the multilinear component of the free Lie algebra with multiple compatible Lie brackets. We introduce a new poset of weighted partitions Π_n^k that allows us to generalize the result. The new poset is a generalization of Π_n and of the poset of weighted partitions Π_n^w introduced by Dotsenko and Khoroshkin and studied by the author and Wachs for the case of two compatible brackets. We prove that the poset Π_n^k with a top element added is EL-shellable and hence Cohen–Macaulay. This and other properties of Π_n^k enable us to answer questions posed by Liu on free multibracketed Lie algebras. In particular, we obtain various dimension formulas and multicolored generalizations of the classical Lyndon and comb bases for the multilinear component of the free Lie algebra. We also obtain a plethystic formula for the Frobenius characteristic of the representation of the symmetric group on the multilinear component of the free multibracketed Lie algebra.

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1. Introduction

Recall that a *Lie bracket* on a vector space V is a bilinear binary product $[\cdot,\cdot] : V \times V \to V$ such that for all $x, y, z \in V$,

$$[x,y] + [y,x] = 0 \qquad \text{(Antisymmetry)}, \tag{1.1}$$

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0 \qquad \text{(Jacobi Identity)}. \tag{1.2}$$

Throughout this paper let $\mathbf{k}$ denote an arbitrary field. The *free Lie algebra on* $[n] := \{1, 2, \dots, n\}$ (over the field $\mathbf{k}$) is the $\mathbf{k}$-vector space generated by the elements of $[n]$ and all the possible bracketings involving these elements subject only to the relations (1.1) and (1.2). Let $\mathcal{L}ie(n)$ denote the *multilinear* component of the free Lie algebra on $[n]$, i.e., the subspace generated by bracketings that contain each element of $[n]$ exactly once. We call these bracketings *bracketed permutations*. For example $[[2,3],1]$ is a bracketed permutation in $\mathcal{L}ie(3)$, while $[[2,3],2]$ is not. For any set S, the symmetric group $\mathfrak{S}_S$ is the group of permutations of S. In particular we denote by $\mathfrak{S}_n := \mathfrak{S}_{[n]}$ the group of permutations of the set $[n]$. The symmetric group $\mathfrak{S}_n$ acts naturally on $\mathcal{L}ie(n)$ making it into an $\mathfrak{S}_n$-module. A permutation $\tau \in \mathfrak{S}_n$ acts on the bracketed permutations by replacing each letter i by $\tau(i)$. For example $(1,2)\ [[[3,5],[2,4]],1] = [[[3,5],[1,4]],2]$. Since this action respects the relations (1.1) and (1.2), it induces a representation of $\mathfrak{S}_n$ on $\mathcal{L}ie(n)$. It is a classical result that

$$\dim \mathcal{L}ie(n) = (n-1)!.$$

Although the $\mathfrak{S}_n$-module $\mathcal{L}ie(n)$ is an algebraic object it turns out that the information needed to completely describe this object is of a combinatorial nature. Let P denote a *partially ordered set* (or *poset* for short). To every poset P one can associate a simplicial complex $\Delta(P)$ (called the *order complex*) whose faces are the chains (totally ordered subsets) of P. Consider now the poset Π_n of set partitions of $[n]$ ordered by refinement. The symmetric group $\mathfrak{S}_n$ acts naturally on Π_n and this action induces isomorphic representations of $\mathfrak{S}_n$ on the unique nonvanishing reduced simplicial homology $\widetilde{H}_{n-3}(\overline{\Pi}_n)$ and cohomology $\widetilde{H}^{n-3}(\overline{\Pi}_n)$ of the order complex $\Delta(\overline{\Pi}_n)$ of the proper part $\overline{\Pi}_n := \Pi_n \setminus \{\hat{0}, \hat{1}\}$ of Π_n. It is a classical result that

$$\mathcal{L}ie(n) \simeq_{\mathfrak{S}_n} \widetilde{H}_{n-3}(\overline{\Pi}_n) \otimes \mathrm{sgn}_n, \tag{1.3}$$

where sgn_n is the sign representation of $\mathfrak{S}_n$.

Equation (1.3) was observed by Joyal [28] by comparing a computation of the character of $\widetilde{H}_{n-3}(\overline{\Pi}_n)$ by Hanlon and Stanley (see [36]), to an earlier formula of Brandt [10] for the character of $\mathcal{L}ie(n)$. Joyal [28] gave a proof of the isomorphism using his theory of species. The first purely combinatorial proof was obtained by Barcelo [2] who provided a bijection between known bases for the two $\mathfrak{S}_n$-modules (Björner's NBC basis for $\widetilde{H}_{n-3}(\overline{\Pi}_n)$ and