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Families of multisums as mock theta functions

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In view of the Bailey lemma and the relations between Hecke-type sums and Appell–Lerch sums given by Hickerson and Mortenson, we find that many Bailey pairs given by Slater can be used to deduce mock theta functions. Therefore, by constructing generalized Bailey pairs with more parameters, we derive some new families of mock theta functions. Meanwhile, some identities between new mock theta functions and classical ones are established. Furthermore, based on the proofs of the main theorems, many q-hypergeometric transformations are obtained.

1. Introduction

Mock theta functions were first introduced by Ramanujan [19, pp. 354–355] in his last letter to G.H. Hardy. Ramanujan listed 17 mock theta functions which were assigned orders 3, 5, and 7. Since then, constructing new mock theta functions has received a great deal of attention. See, for example, [2,5,9,17]. Around the year 2000, connections between mock theta functions and Maass forms brought a new insight for the study of mock theta functions. Based on the work of Zagier [23], Zwegers [25], and Bringmann

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and Ono [7,8], we know that each of Ramanujan's mock theta functions is the holomorphic part of a weight 1/2 harmonic weak Maass form with a weight 3/2 unary theta function as its shadow. More importantly, Zagier [24] and Zwegers [25] showed that specializations of Appell–Lerch sums give mock theta functions. For these developments, see Ono's memoir [18]. Recently, Hickerson and Mortenson [11] built some relations between Hecke-type sums and Appell–Lerch sums. By means of these relations and the Bailey lemma, Lovejoy and Osburn [13,14,16] and Lovejoy [12] found some new families of mock theta functions. The motivation for this paper is an observation that many Bailey pairs given by Slater [20,21] can be used to construct new mock theta functions. Based on the forms of these Bailey pairs, we derive more families of q-hypergeometric multisums as mock theta functions by establishing generalized Bailey pairs with more parameters.

Here we follow the standard q-series notations

$$(a)_n := (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \quad \text{and} \quad (a_1,a_2,\ldots,a_m;q)_n := \prod_{j=1}^{m}(a_j;q)_n,$$

where $|q|<1$ and $n \in \mathbb{N}\cup\{\infty\}$.

Appell–Lerch sums which were first studied by Appell [6] are stated as follows:

$$m(x,q,z) := \frac{1}{j(z;q)}\sum_{r\in\mathbb{Z}}\frac{(-1)^r q^{\binom{r}{2}} z^r}{1-q^{r-1}xz},$$

where $x,z \in \mathbb{C}^* := \mathbb{C}/\{0\}$ with neither z nor xz an integral power of q. Here

$$j(z;q) := (z,q/z,q;q)_\infty \quad \text{and} \quad j(z_1,z_2,\cdots,z_m;q) := \prod_{i=1}^{m} j(z_i;q).$$

Notice that

$$j(zq^n;q) = (-1)^n z^{-n} q^{-\binom{n}{2}} j(z;q), \quad n\in\mathbb{Z}. \tag{1.1}$$

For convenience, we also define that for $a\in\mathbb{Z}$ and $m\in\mathbb{N}$,

$$J_{a,m} := j(q^a;q^m), \quad \overline{J}_{a,m} := j(-q^a;q^m), \quad \text{and} \quad J_m := J_{m,3m} = (q^m;q^m)_\infty.$$

Hecke-type sums which were given by Hecke [10] are defined as

$$f_{a,b,c}(x,y,q) := \sum_{sg(n)=sg(j)} sg(n)(-1)^{n+j}x^n y^j q^{a\binom{n}{2}+bnj+c\binom{j}{2}},$$

where $x,y\in\mathbb{C}^* := \mathbb{C}/\{0\}$, $sg(n) := 1$ for $n\geq 0$, and $sg(n) := -1$ for $n<0$.

The Bailey lemma plays a very important role in the study of mock theta functions. A Bailey pair relative to (a,q) is a pair of sequences $(\alpha_n,\beta_n)_{n\geq 0}$ satisfying