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On the Lyndon dynamical system

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ABSTRACT

Given a totally finite ordered alphabet $\mathcal{A}$, endowing the set of words over $\mathcal{A}$ with the alternating lexicographic order (see [6]), we define a new class of Lyndon words. We study the fundamental properties of the associated symbolic dynamical systems called Lyndon system. We derive some fundamental properties of the beta-shift with negative base by relating it with the Lyndon system. We find, independently of W. Steiner's method (see [11]), the conditions for which a word is the $(-\beta)$-expansion of $-\frac{\beta}{\beta+1}$ for some $\beta > 1$.

1. Introduction

In the areas of combinatorics and computer sciences, a Lyndon word is a word which is lexicographically less than all its permutations. Roger Lyndon introduced them in 1954 on the standard name "lexicographic sequences". He used them to construct a basis for the homogeneous part of a given degree in Lie algebra (see [8,2]).

By definition, we call Lyndon word $x_1x_2x_3\cdots$ over a totally ordered alphabet all word which is lexicographically less than all of its suffixes.

$$x_1x_2\cdots x_{n-1}x_n\cdots \leq_{lex} x_kx_{k+1}\cdots x_nx_{n+1}\cdots .$$

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Consider a finite ordered alphabet $\mathcal{A} = \{0, 1, \cdots, d\}$. We denote by $\mathcal{A}^{\mathbb{N}}$ the set of finite and infinite words over $\mathcal{A}$. Endowing $\mathcal{A}^{\mathbb{N}}$ with the alternate order on words (see [10,6]), we can define Lyndon word using this new order. We call them alternate Lyndon words. The definition given in [10] generalizes Lyndon words. To each Lyndon word we attach a symbolic dynamical system. If the entropy of such a system is positive (shall we say $\log \beta > 0$), we show that the associated alternate Lyndon word is a $(-\beta)$-representation of $-\dfrac{\beta}{\beta+1}$ (Proposition 3). When β tends to 1, the alternate Lyndon word tends to

$$\phi^{\infty}(1) = 10011100100100111001110011100100100111001 00 \cdots,$$

where ϕ is a morphism on $\{0, 1\}$ such that $\phi(0) = 1$ and $\phi(1) = 100$ (Theorem 1). We establish a link with expansions in negative bases. Indeed, for a fixed real $\beta > 1$, we can see the alternate order as a tool of controllability of the representations of numbers in base $-\beta$. In the negative expansion pioneering paper [6], it is proved that for some $\beta > 1$, the $(-\beta)$-expansion of $-\frac{\beta}{\beta+1}$, $(d_i)_{i\geq 1}$ is such that all sub-word of a $(-\beta)$-expansion $(x_i)_{i\geq 1}$ is greater than $(d_i)_{i\geq 1}$ in the sense of the alternate order. In particular, $(d_i)_{i\geq 1}$ is less than all of its sub-words (in the sense of the alternate order). Thanks to this link, we give the necessary and sufficient conditions for which a word over a finite totally ordered alphabet can be the $(-\beta)$-expansion of $-\frac{\beta}{\beta+1}$ for some $\beta > 1$ (Theorem 3).

1.1. Definitions and generality

Definition 1. Let $\mathcal{A}$ be a totally ordered alphabet endowed with an order "$<$". We call *Lyndon word* over $\mathcal{A}$ with respect to the order "$<$", all word $x_1x_2x_3\cdots$ such that

$$x_1x_2x_3\cdots \leq x_nx_{n+1}x_{n+2}\cdots, \ \forall n, \tag{1}$$

with x_i in $\mathcal{A}$ for all i.

- The word $x_1x_2x_3\cdots$ is said to be *strong Lyndon word* if in (1) all inequalities are strict.
- The Lyndon word $x_1x_2\cdots$ is weak if in (1) equality holds for some n.

$$x_1x_2x_3x_4\cdots = x_nx_{n+1}x_{n+2}\cdots.$$

Then, the weak Lyndon words over the alphabet $\mathcal{A}$ are periodic.

1.2. Ito and Sadahiro order

Let $\mathcal{A} = \{0, 1, \cdots, d\}$ be an alphabet. Consider two words $x_1x_2\cdots x_n$ and $y_1y_2\cdots y_n$ on $\mathcal{A}$. We will say $x_1x_2\cdots x_n$ is less than $y_1y_2\cdots y_n$ in the sense of alternate order (and we note $x_1x_2\cdots x_n \prec y_1y_2\cdots y_n$) if there exists an integer $k \leq n$ such that for all $i < k$,