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Parabolic Kazhdan–Lusztig polynomials for quasi-minuscule quotients

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ABSTRACT

We study the parabolic Kazhdan–Lusztig polynomials for the quasi-minuscule quotients of Weyl groups. We give explicit closed combinatorial formulas for the parabolic Kazhdan–Lusztig polynomials of type q. Our study implies that these are always either zero or a monic power of q, and that they are not combinatorial invariants. We conjecture a combinatorial interpretation for the parabolic Kazhdan–Lusztig polynomials of type -1.

1. Introduction

In 1979, Kazhdan and Lusztig [16] introduced a family of polynomials, indexed by pairs of elements in a Coxeter group W, which plays an important role in various areas

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of mathematics, including the algebraic geometry and topology of Schubert varieties and representation theory (see, e.g., [1] p. 171 and the references cited there). These celebrated polynomials are now known as the Kazhdan–Lusztig polynomials of W (see, e.g., [1] or [14]). In 1987, Deodhar [7] developed an analogous theory for the parabolic setup. Given any parabolic subgroup W_J in a Coxeter system (W, S), Deodhar introduced two Hecke algebra modules (one for each of the two roots q and -1 of the polynomial $x^2-(q-1)x-q$) and two families of polynomials $\{P^{J,q}_{u,v}(q)\}_{u,v\in W^J}$ and $\{P^{J,-1}_{u,v}(q)\}_{u,v\in W^J}$ indexed by pairs of elements of the set of minimal coset representatives W^J. These polynomials are the parabolic analogues of the Kazhdan–Lusztig polynomials: while they are related to their ordinary counterparts in several ways (see, e.g., §2 and [7], Proposition 3.5), they also play a direct role in several areas such as the geometry of partial flag manifolds [15], the theory of Macdonald polynomials [12,13], tilting modules [24,25], generalized Verma modules [5], canonical bases [10,28], the representation theory of the Lie algebra $\mathfrak{gl}_n$ [20], quantized Schur algebras [29], quantum groups [8], and physics (see, e.g., [11], and the references cited there). The computation of these polynomials is a very difficult task. Although a geometric interpretation for the (ordinary and parabolic) polynomials exists (see [17] and [15]) in the case of Weyl groups and an algebraic interpretation exists for the ordinary ones [9] for all Coxeter systems, there are very few explicit formulas for them (see, e.g., [1], p. 172, and the references cited there).

The purpose of this work is to study the parabolic Kazhdan–Lusztig polynomials for the quasi-minuscule quotients of Weyl groups. These quotients possess noteworthy combinatorial and geometric properties (see, e.g., [18] and [27]). The parabolic Kazhdan–Lusztig polynomials for the minuscule quotients have been computed in [19,2–4]. In this work we turn our attention to the quasi-minuscule quotients that are not minuscule (also known as (co-)adjoint quotients). More precisely, we obtain closed combinatorial formulas for the parabolic Kazhdan–Lusztig polynomials of type q of these quotients for the classical Weyl groups. Our results imply that these are always either zero or a monic power of q for all quasi-minuscule quotients, and that they are not combinatorial invariants. For the parabolic Kazhdan–Lusztig polynomials of type -1 we conjecture explicit combinatorial interpretations.

The organization of the paper is as follows. In Section 2 we recall some definitions, notation and results that are used in the sequel. In Section 3 we give combinatorial descriptions of the quasi-minuscule quotients of classical Weyl groups. In Section 4 we give combinatorial formulas for the parabolic Kazhdan–Lusztig polynomials of type q of (co-)adjoint quotients of classical Weyl groups. Our results imply that these polynomials are always either zero or a monic power of q for all quasi-minuscule quotients, and that they are not combinatorial invariants. In Section 5 we derive some consequences of our results for the classical Kazhdan–Lusztig polynomials. Finally, in Section 6 we present our conjectured combinatorial interpretations for the parabolic Kazhdan–Lusztig polynomials of type -1 of the (co-)adjoint quotients of classical Weyl groups, and the evidence that we have in their favor.