Advances in Applied Mathematics 78 (2016) 94–113

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Asymptotic normality and combinatorial aspects of the prefix exchange distance distribution

Simona Grusea [a,*], Anthony Labarre [b]

[a] *Institut de Mathématiques de Toulouse, INSA de Toulouse, Université de Toulouse, France*
[b] *Université Paris-Est, LIGM UMR CNRS 8049, Université Paris-Est Marne-la-Vallée, 5 boulevard Descartes, 77420 Champs-sur-Marne, France*

A R T I C L E I N F O

Article history:
Received 12 February 2016
Received in revised form 7 April 2016
Accepted 7 April 2016
Available online 21 April 2016

MSC:
68R05
05A05
05A15
60F05

Keywords:
Star poset
Whitney numbers
Combinatorial proofs
Permutation
Distance
Prefix exchange
Distribution
Asymptotic normality

A B S T R A C T

The prefix exchange distance of a permutation is the minimum number of exchanges involving the leftmost element that sorts the permutation. We give new combinatorial proofs of known results on the distribution of the prefix exchange distance for a random uniform permutation. We also obtain expressions for the mean and the variance of this distribution, and finally, we show that the normalised prefix exchange distribution converges in distribution to the standard normal distribution.

* Corresponding author.
E-mail addresses: grusea@insa-toulouse.fr (S. Grusea), Anthony.Labarre@u-pem.fr (A. Labarre).

http://dx.doi.org/10.1016/j.aam.2016.04.002
0196-8858/

1. Introduction

An ever-growing body of research has been devoted to the study of various measures of disorder on permutations, with the intention of expressing how many elementary operations (whose type may vary but which are fixed beforehand) they should undergo in order to become sorted. One of the earliest examples of such a measure is the *Cayley distance*, which corresponds to the minimum number of transpositions that must be applied to a permutation in order to obtain the identity permutation. This distance is easily expressed in terms of the number of *cycles* of the permutation [6], and the *signless Stirling numbers of the first kind* can be used to characterise exactly the distribution of the Cayley distance — i.e., the number of permutations of n elements with Cayley distance k. Motivations for studying these distances and their distributions outside pure mathematical fields include the study of sorting algorithms [10], genome comparison [11], and the design of interconnection networks [16].

We focus in this paper on the *prefix exchange* operation, a restricted kind of transposition that swaps any element of a permutation with its first element. This operation was introduced by Akers and Krishnamurthy [1], who also gave a formula for computing the associated *prefix exchange distance*, i.e., the minimum number of prefix exchanges required to transform a given permutation into the identity permutation. Portier and Vaughan [18] later succeeded in obtaining the generating function of the corresponding distribution, which they then used to derive an explicit formula (with subsequent corrections by Shen and Qiu [21]) as well as recurrence formulas for computing the so-called "Whitney numbers of the second kind for the star poset", i.e., the number of permutations of size n with prefix exchange distance k (see Portier [17] for a table with the first few terms).

We revisit in this paper the results obtained by Portier and Vaughan [18] by taking the opposite direction: we first obtain new proofs for their exact and recurrence formulas, and then use these formulas to recover their expression for the generating function. Our proofs are purely combinatorial, a desirable property since such proofs are often simpler in addition to providing new insight into the underlying objects [5,22].

We then proceed to obtaining the mean and the variance of the distribution, and finally, we examine the behaviour of this distribution when n tends to infinity: in particular, we show that the normalised prefix exchange distribution converges in distribution to the standard normal distribution. Our result enriches the family of combinatorial sequences which were previously shown to behave asymptotically normally, like the (signless) Stirling numbers of first and second kind, the Eulerian numbers, the adjacent transposition distance distribution and the related distribution of the number of inversions in a permutation — for precise definitions of these sequences and asymptotic normality results, as well as other examples, see [9,3,12].