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Differential (Lie) algebras from a functorial point of view

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ABSTRACT

It is well-known that any associative algebra becomes a Lie algebra under the commutator bracket. This relation is actually functorial, and this functor, as any algebraic functor, is known to admit a left adjoint, namely the universal enveloping algebra of a Lie algebra. This correspondence may be lifted to the setting of differential (Lie) algebras. In this contribution it is shown that, also in the differential context, there is another, similar, but somewhat different, correspondence. Indeed any commutative differential algebra becomes a Lie algebra under the Wronskian bracket $W(a,b) = ab' - a'b$. It is proved that this correspondence again is functorial, and that it admits a left adjoint, namely the differential enveloping (commutative) algebra of a Lie algebra. Other standard functorial constructions, such as the tensor and symmetric algebras, are studied for algebras with a given derivation.

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1. Introduction and motivations

Given a commutative (associative) ring R with a unit, a *Lie R-algebra* (or just a *Lie algebra*) is a pair $(g, [-,-])$ where g is a R-module and $[-,-]: g \times g \to g$ is a *Lie bracket* (or just a *bracket*), i.e., a R-bilinear map which is

- *alternating*: this means that $[x,x] = 0$ for every $x \in g$,
- and satisfies the *Jacobi identity*: for every $x, y, z \in g$,

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0.$$

A Lie algebra is said to be *commutative* when its bracket is the zero bracket, i.e., $[x,y] = 0$ for every x, y. Thus it is essentially only a R-module.

Lie algebras are very common in the context of associative algebras since any (say unital) associative R-algebra $(A,*)$ may be turned into a Lie algebra when it is equipped with the so-called *commutator bracket*

$$[x,y] = x * y - y * x,$$

$x, y \in A$. The Lie algebra $(A, [-,-])$ is then referred to as the *underlying Lie algebra* of $(A,*)$. Because any algebra homomorphism induces a homomorphism of Lie algebras between the underlying Lie algebras, this correspondence between (unital) associative algebras and Lie algebras is actually functorial. Furthermore this functor is well-known to admit a left adjoint, namely the *universal enveloping algebra* $\mathsf{U}(g,[-,-])$ of a Lie algebra $(g,[-,-])$. It is given by $\mathsf{U}(g,[-,-]) = \mathsf{T}(g)/I$ where I is the two-sided ideal of the tensor R-algebra $\mathsf{T}(g)$ on the R-module g generated by $x \otimes y - y \otimes x - [x,y]$, $x, y \in g$ (for more details see e.g. [6]).

This algebra is *universal* among all (unital) algebras with a Lie map from $(g,[-,-])$ to its underlying Lie algebra. By this is meant the following. Let j_g be the composite R-linear map $g \hookrightarrow \mathsf{T}(g) \xrightarrow{\pi} \mathsf{U}(g,[-,-])$, where the first arrow is the canonical inclusion from the R-module g into its tensor algebra, and π is the canonical projection. The linear map j_g happens to be a homomorphism of Lie algebras from $(g,[-,-])$ to the underlying Lie algebra of $\mathsf{U}(g,[-,-])$. Now, given any other associative algebra $(B,*)$ with a unit and a homomorphism $\phi: (g,[-,-]) \to (B,[-,-])$ of Lie algebras, then ϕ uniquely factors through j_g, i.e., there exists a unique homomorphism of algebras $\hat{\phi}: (A,*) \to (B,*)$ such that the following diagram commutes (in the category of Lie algebras).

$$\begin{tikzcd} (g,[-,-]) \arrow[r, "\phi"] \arrow[d, "j_g"'] & B \\ \mathsf{U}(g,[-,-]) \arrow[ur, "\hat{\phi}"'] & \end{tikzcd} \qquad (1)$$