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Dimension polynomials of difference local algebras

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ABSTRACT

We extend the concept of the difference dimension polynomial of a difference field extension to difference local algebras. The main theorem of the paper establishes the existence and form of the dimension polynomial associated with the localization of a finitely generated well-mixed difference algebra at a prime reflexive difference ideal.

1. Introduction

The role of difference dimension polynomials in the study of algebraic difference equations is similar to the role of Hilbert polynomials in commutative algebra and algebraic geometry, as well as to the role of differential dimension polynomials in differential algebra. The notion of a difference dimension polynomial was introduced in [4], practically all known results on such polynomials (including algorithms for their computation) can be found in [3] and [5]. In particular, as it is shown in [5, Chapter 7], the difference dimen-

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sion polynomial of a system of algebraic difference equations expresses the A. Einstein's strength of the system. This fact determines the importance of the study of difference dimension polynomials for the qualitative theory of algebraic difference equations.

Another important application of difference dimension polynomials is based on the fact that if P is a prime reflexive difference ideal of a finitely generated difference algebra $R = K\{\eta_1, \dots, \eta_n\}$ over a difference field K, then the quotient field of R/P is a difference field extension of K generated by the images of η_i in R/P. The difference dimension polynomial of this extension, therefore, characterizes the ideal P; assigning such polynomials to prime reflexive difference ideals has led to a number of new results on the Krull-type dimension of difference algebras (see, for example, [6] and [5, Section 4.6]).

In this paper, we introduce a dimension polynomial associated with the localization of a finitely generated well-mixed difference algebra at a reflexive difference prime ideal. Our main result (Theorem 2) establishes the existence of such a polynomial and gives its expression in terms of difference dimension polynomials of certain finitely generated difference field extensions. Note that the latter polynomials can be computed using the techniques developed in [3] and [5].

2. Preliminaries

Throughout the paper, $\mathbb{N}$ denotes the set of all nonnegative integers. By a ring we always mean an associative ring with unity denoted by 1. If S is a subset of a ring R, then the ideal of R generated by the set S is denoted by (S).

Recall that a *difference ring* is a commutative ring R considered together with a set $\sigma = \{\alpha_1, \dots, \alpha_m\}$ of mutually commuting ring endomorphisms $\alpha_i : R \to R$ called *translations*. The set σ is called a *basic set* of the difference ring R. In what follows, we assume that the translations are injective (this assumption is standard in most works on difference algebra). Furthermore, we will consider the free commutative semigroup T_σ of all power products $\tau = \alpha_1^{k_1} \dots \alpha_m^{k_m}$ $(k_1, \dots, k_m \in \mathbb{N})$.

If R is a difference ring and R_0 a subring of R such that $\alpha_i(R_0) \subseteq R_0$ for every translation α_i, then R_0 is called a *difference subring* of R. In this case, the restriction of the translation on R_0 is denoted by the same symbol α_i. If J is an ideal of R such that $\alpha(J) \subseteq J$ for all $\alpha \in \sigma$, then J is called a *difference* ideal of R. If, in addition, the inclusion $\alpha_i(a) \in J$ implies $a \in J$ for any $a \in R$, $1 \le i \le m$, then the difference ideal J is called *reflexive*. In this case, the factor ring R/J has a natural structure of a difference ring with the same basic set σ where $\alpha(a + J) = \alpha(a) + J$ for any coset $a + J \in R/J$ and $\alpha \in \sigma$. If a difference ideal is prime, it is referred to as a *prime difference ideal*. If a difference ring is a field, it is called a *difference field*.

If S is a subset of R, then the intersection of all difference ideals of R containing S is denoted by $[S]$. Clearly, $[S]$ is the smallest difference ideal of R containing S; as an ideal, it is generated by the set $\{\tau(a) \,|\, a \in S, \tau \in T_\sigma\}$.

If R_0 is a difference subring of a difference ring R and $S \subseteq R$, then the intersection of all difference subrings of R containing R_0 and S (that is, the smallest difference