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A decision method for the integrability of differential–algebraic Pfaffian systems ☆

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We prove an effective integrability criterion for differential–algebraic Pfaffian systems leading to a decision method of consistency with a triple exponential complexity bound. As a byproduct, we obtain an upper bound for the order of differentiations in the differential Nullstellensatz for these systems.

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1. Introduction

Let $\mathbf{x} := x_1, \ldots, x_m$ and $\mathbf{y} := y_1, \ldots, y_n$ be two sets of variables; the first ones represent the independent variables (i.e. those defining the partial derivations) and the second ones are considered as differential unknowns.

The notion of *Pfaffian system* is introduced by J.F. Pfaff in [22] and in its simplest form it is defined as a system of partial differential equations of the type:

$$\Sigma := \left\{ \frac{\partial y_i}{\partial x_j} = f_{ij}(\mathbf{x}, \mathbf{y}) \quad i = 1, \ldots, n, \; j = 1, \ldots, m, \right. \tag{1}$$

where each f_{ij} is an analytic function around a certain fixed point $(\mathbf{x}_0, \mathbf{y}_0) \in \mathbb{C}^m \times \mathbb{C}^n$.

The properties of these systems were extensively studied during the XIXth century by notable mathematicians as Jacobi [13], Clebsch [2] and Frobenius [6] (see also [11] for a detailed historical approach). One of the main problems considered is the so-called *complete integrability of a Pfaffian system* Σ: the existence of a neighborhood U of the point $(\mathbf{x}_0, \mathbf{y}_0)$ such that for every point $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) \in U$ there exists a solution γ of Σ such that $\gamma(\widehat{\mathbf{x}}) = \widehat{\mathbf{y}}$. A complete solution of this problem is known today as the "Frobenius Theorem" (see [6]):

Frobenius Theorem. *Let Σ be the Pfaffian system (1). Then Σ is completely integrable in $(\mathbf{x}_0, \mathbf{y}_0)$ if and only if for all indices i, j, k with $i = 1, \ldots, n$, and $j, k = 1, \ldots, m$, the function $D_j(f_{ik}) - D_k(f_{ij})$ vanishes in a neighborhood of $(\mathbf{x}_0, \mathbf{y}_0)$.*

Here, $D_j(h)$ denotes the j-th total derivative with respect to Σ of any analytic function h in the variables $(\mathbf{x}, \mathbf{y})$, namely $D_j(h) := \dfrac{\partial h}{\partial x_j} + \displaystyle\sum_{i=1}^{n} \dfrac{\partial h}{\partial y_i} f_{ij}$. Observe that the vanishing of the functions $D_j(f_{ik}) - D_k(f_{ij})$ is clearly a necessary condition because of the equality of the mixed derivatives of an analytic function, but the sufficiency is not obvious.

We remark that Frobenius Theorem is more general than the statement above because it remains true also for systems of differential linear 1-forms (today called *Pfaffian forms*). In this sense Frobenius's article may be considered as a main source of inspiration for the transcendental work by Cartan about integrability of exterior differential systems developed in the first half of the XXth century (see [1]).

In Cartan's theory two notions play a main rôle: the *prolongation* and the *involutivity* of an exterior differential system. In the particular case of differential equation systems these notions can be easily paraphrased. The prolongation of a system consists simply in applying derivations to it. On the other hand, an involutive system is a system with no hidden integrability conditions; in other words, no prolongation is necessary to find new constraints. Roughly speaking, the involutive systems are those to which the classical method of resolution by means of a power series with indeterminate coefficients (known as Frobenius method) can be applied in order to search for a solution or to decide that the system is not integrable.