Advances in Applied Mathematics 72 (2016) 215–230

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

# Superstability and central extensions of algebraic groups

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A R T I C L E   I N F O

*Article history:*
Received 29 December 2014
Received in revised form 4 September 2015
Accepted 4 September 2015
Available online 19 September 2015

*MSC:*
03C45
03C60
19C09
03C98

*Keywords:*
Differential algebra
Model theory
Differential algebraic groups
Superstable groups
Model theory of differential fields
Central extensions

A B S T R A C T

Altinel and Cherlin proved that any perfect central extension of a simple algebraic group over an algebraically closed field which happens to be of finite Morley rank is actually a finite central extension and is itself an algebraic group. We will prove an infinite rank version of their result with an additional hypothesis, while giving an example which shows the necessity of this hypothesis. The inspiration for the work comes from differential algebra; namely, a differential algebraic version of the results here was used by the second author to answer a question of Cassidy and Singer. The work here also provides an alternate path to the same answer.

An almost simple superstable group $G$ is one in which every definable normal subgroup $H$ has the property that $RU(G) > RU(H) \cdot n$ for any $n \in \mathbb{N}$. We prove that any almost simple superstable group which is a central extension of a simple algebraic group is actually a finite central extension and is an algebraic group. We also explain the applications of this result to differential algebraic groups. Many of the central ideas of the proof of our main theorem are an adaptation of techniques

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http://dx.doi.org/10.1016/j.aam.2015.09.002

developed by the second author in the setting of differential algebraic groups.



## 1. Introduction

Cherlin conjectured that every simple $\omega$-stable group is an algebraic group over an algebraically closed field [10]; this is a special case of a conjecture of Berline [5] with the weaker hypothesis of superstability in place of $\omega$-stability. In particular, this conjecture implies that every simple $\omega$-stable (more generally superstable) group is of finite Morley rank in the language of groups, and it also implies the more widely cited (sometimes called Cherlin–Zilber) conjecture that every simple group of finite Morley rank is an algebraic group. In [2], Altinel and Cherlin proved:

**Theorem 1.1.** *Let $G$ be perfect group of finite Morley rank such that $G/Z(G)$ is a quasisimple algebraic group. Then $G$ is an algebraic group and $Z(G)$ is finite.*

**Remark 1.2.** In this article, $Z(G)$ denotes the center of $G$. Quasisimple will be taken to mean an infinite group with no nontrivial connected normal closed subgroups; this notion and some related notions will be considered for groups in various categories (see Section 2).

This generalized previous work [1] in which the theorem was proved with an additional *tameness* assumption, which simplified the argument. Part of the difficulty with applying this theorem in differential algebra is that the finite Morley rank assumption is often too strong. We will give a specific example of the difficulty of the situation later in the introduction.

In the opening lines of [2], the authors note that universal central extensions of simple algebraic groups (in the category of abstract groups) are not always of finite type, so some assumption about the central extension is required for Theorem 1.1. In Section 4 we will show a counterexample to the statement of Theorem 1.1 in the general $\omega$-stable setting (without the assumption of finite rank), which essentially shows that some hypothesis on the rank of the center is required.

Before the work of Cherlin, Altinel and Borovik, Pillay and Sokolovic [20, Theorem 10] proved a related theorem:

**Theorem 1.3.** *Let $\mathcal{M}$ be an $\omega$-stable structure. Let $K$ be an algebraically closed field definable in $\mathcal{M}$ and $H$ a quasisimple algebraic group over $K$. Let $G$ be a definably quasisimple group definable in $\mathcal{M}$ such that there is a definable homomorphism $f : G \to H$ which is onto. Then there is an algebraic group $H_1$ over $K$ and a definable isomorphism $G \to H_1$.*