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Families of multiweights and pseudostars

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Let $\mathcal{T} = (T, w)$ be a weighted finite tree with leaves $1, \dots, n$. For any $I := \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, let $D_I(\mathcal{T})$ be the weight of the minimal subtree of T connecting $i_1, \dots, i_k$; the $D_I(\mathcal{T})$ are called k-weights of $\mathcal{T}$. Given a family of real numbers parametrized by the k-subsets of $\{1, \dots, n\}$, $\{D_I\}_{I \in \binom{\{1,\dots,n\}}{k}}$, we say that a weighted tree $\mathcal{T} = (T, w)$ with leaves $1, \dots, n$ realizes the family if $D_I(\mathcal{T}) = D_I$ for any I.
In [14] Pachter and Speyer proved that, if $3 \leq k \leq (n+1)/2$ and $\{D_I\}_{I \in \binom{\{1,\dots,n\}}{k}}$ is a family of positive real numbers, then there exists at most one positive-weighted essential tree $\mathcal{T}$ with leaves $1, \dots, n$ that realizes the family (where "essential" means that there are no vertices of degree 2). We say that a tree P is a pseudostar of kind (n, k) if the cardinality of the leaf set is n and any edge of P divides the leaf set into two sets such that at least one of them has cardinality $\geq k$. Here we show that, if $3 \leq k \leq n-1$ and $\{D_I\}_{I \in \binom{\{1,\dots,n\}}{k}}$ is a family of real numbers realized by some weighted tree, then there is exactly one weighted essential pseudostar $\mathcal{P} = (P, w)$ of kind (n, k) with leaves $1, \dots, n$ and without internal edges of weight 0, that realizes the family; moreover we describe how any other weighted tree realizing the family can be obtained

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from $\mathcal{P}$. Finally we examine the range of the total weight of the weighted trees realizing a fixed family.

1. Introduction

For any graph G, let $E(G)$, $V(G)$ and $L(G)$ be respectively the set of the edges, the set of the vertices and the set of the leaves of G. A **weighted graph** $\mathcal{G} = (G, w)$ is a graph G endowed with a function $w : E(G) \to \mathbb{R}$. For any edge e, the real number $w(e)$ is called the weight of the edge. If all the weights are nonnegative (respectively positive), we say that the graph is **nonnegative-weighted** (respectively **positive-weighted**); if the weights of the internal edges are nonzero, we say that the graph is **internal-nonzero-weighted** and, if all the weights are nonnegative and the ones of the internal edges are positive, we say that the graph is **internal-positive-weighted**. For any finite subgraph G' of G, we define $w(G')$ to be the sum of the weights of the edges of G'. In this paper we will deal only with weighted finite trees.

Definition 1. Let $\mathcal{T} = (T, w)$ be a weighted tree. For any distinct $i_1, \ldots, i_k \in V(T)$, we define $D_{\{i_1,\ldots,i_k\}}(\mathcal{T})$ to be the weight of the minimal subtree whose vertex set contains $i_1, \ldots, i_k$. We call this subtree "the subtree realizing $D_{\{i_1,\ldots,i_k\}}(\mathcal{T})$". More simply, we denote $D_{\{i_1,\ldots,i_k\}}(\mathcal{T})$ by $D_{i_1,\ldots,i_k}(\mathcal{T})$ for any order of $i_1, \ldots, i_k$. We call the $D_{i_1,\ldots,i_k}(\mathcal{T})$ the k**-weights** of $\mathcal{T}$ and we call a k-weight of $\mathcal{T}$ for some k a **multiweight** of $\mathcal{T}$.

If S is a subset of $V(T)$, the k-weights give a vector in $\mathbb{R}^{\binom{S}{k}}$. This vector is called k**-dissimilarity vector** of $(\mathcal{T}, S)$. Equivalently, we can speak of the **family of the** k**-weights** of $(\mathcal{T}, S)$.

If S is a finite set, $k \in \mathbb{N}$ and $k < \#S$, we say that a family of real numbers $\{D_I\}_{I \in \binom{S}{k}}$ is **treelike** (respectively p-treelike, nn-treelike, inz-treelike, ip-treelike) if there exist a weighted (respectively positive-weighted, nonnegative-weighted, internal-nonzero-weighted, internal-positive-weighted) tree $\mathcal{T} = (T, w)$ and a subset S of the set of its vertices such that $D_I(\mathcal{T}) = D_I$ for any k-subset I of S. If in addition $S \subset L(T)$, we say that the family is **l-treelike** (respectively p-l-treelike, nn-l-treelike, inz-l-treelike, ip-l-treelike).

Graphs and in particular weighted graphs may have applications in several disciplines, such as biology, psychology, archeology, engineering. Phylogenetic trees are positive-weighted trees whose vertices represent species and the weight of an edge is given by how much the DNA sequences of the species represented by the vertices of the edge differ. There is a wide literature concerning graphlike dissimilarity families and treelike dissimilarity families, in particular concerning methods to reconstruct positive-weighted trees from their dissimilarity families; these methods are used by biologists to reconstruct phylogenetic trees (see for example [13,19] and [8,17] for overviews); also archeologists