Advances in Applied Mathematics 76 (2016) 1–33

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Reconstruction of convex bodies from surface tensors

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ARTICLE INFO

Article history:
Received 24 June 2015
Received in revised form 4 January 2016
Accepted 4 January 2016
Available online 26 January 2016

MSC:
52A10
52A20
44A60
60D05

Keywords:
Convex body
Shape
Reconstruction algorithm
Surface tensor
Harmonic intrinsic volume
Generalized Wirtinger's inequality

ABSTRACT

We present two algorithms for reconstruction of the shape of convex bodies in the two-dimensional Euclidean space. The first reconstruction algorithm requires knowledge of the exact surface tensors of a convex body up to rank s for some natural number s. When only measurements subject to noise of surface tensors are available for reconstruction, we recommend to use certain values of the surface tensors, namely harmonic intrinsic volumes instead of the surface tensors evaluated at the standard basis. The second algorithm we present is based on harmonic intrinsic volumes and allows for noisy measurements. From a generalized version of Wirtinger's inequality, we derive stability results that are utilized to ensure consistency of both reconstruction procedures. Consistency of the reconstruction procedure based on measurements subject to noise is established under certain assumptions on the noise variables.

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http://dx.doi.org/10.1016/j.aam.2016.01.001

1. Introduction

The problem of determining and reconstructing an unknown geometric object from indirect measurements is treated in a number of papers, see, e.g., [6]. In [19], a convex body is reconstructed from measurements of its support function. Measurements of the brightness function are used in [7], and in [4] it is shown that a convex body can be uniquely determined up to translation from measurements of its lightness function. Milanfar et al. [17] developed a reconstruction algorithm for planar polygons and quadrature domains from moments of the Lebesgue measure restricted to these sets. In particular, they showed that a non-degenerate convex polygon in $\mathbb{R}^2$ with k vertices is uniquely determined by its moments up to order $2k-3$. The reconstruction algorithm and the uniqueness result were generalized to convex polytopes in $\mathbb{R}^n$ by Gravin et al. in [10].

In continuation of the work in this area, we discuss reconstruction of convex bodies from a certain type of Minkowski tensors. In recent years, Minkowski tensors have been studied intensively. On the applied side, Minkowski tensors have been established as robust and versatile descriptors of shape and morphology of spatial patterns of physical systems, see e.g., [2,22] and the references given there. The importance of Minkowski tensors is further indicated by Alesker's characterization theorem, see [1], that states that products of Minkowski tensors and powers of the metric tensor span the space of tensor-valued valuations on convex bodies satisfying some natural conditions.

In the present work, we consider translation invariant Minkowski tensors, $\Phi_j^s(K)$ of rank s, which are tensors derived from the jth area measure $S_j(K,\cdot)$ of a convex body $K \subseteq \mathbb{R}^n$, $j = 0, \dots, n-1$. For details, see Sections 2 and 3. For a given $j = 1, \dots, n-1$, the set $\{\Phi_j^s(K) \mid s \in \mathbb{N}_0\}$ of all Minkowski tensors determines K up to translation. Calling the equivalence class of all translations of K the *shape* of K, we can say that $\{\Phi_j^s(K) \mid s \in \mathbb{N}_0\}$ determines the shape of K. When only Minkowski tensors $\Phi_j^s(K)$, $s \le s_o$ up to a certain rank s_o are given, this is, in general, no longer true. We establish a stability result (Theorem 4.9) stating that the shapes of two convex bodies are close to one another when the two convex bodies have coinciding Minkowski tensors $\Phi_1^s(K)$ of rank $s \le s_o$. The proof uses a generalization of Wirtinger's inequality (Corollary 4.7), which is different from existing generalizations in the literature (e.g. [5,8]) as it involves a higher order spherical harmonic expansion. We also show (Theorem 4.1) that there always exists a convex polytope P with the same surface tensors $\Phi_{n-1}^s(P)$ of rank $s \le s_o$ as a given convex body. The number of facets of P can be bounded by a polynomial of s_o of degree $n-1$. Using this result, we conclude (Corollary 4.2) that a convex body K is a polytope if the shape of K is uniquely determined by a finite number of surface tensors. In fact, the shape of a convex body K is uniquely determined by a finite number of its surface tensors if and only if K is a polytope (Theorem 4.3).

For actual reconstructions, we restrict considerations to the planar case. We consider two cases. Firstly, the case when the exact tensors are given, and secondly, the case when certain values of the tensors are measured with noise. *Algorithm Surface Tensor*