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Formulas and identities involving the Askey–Wilson operator

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A B S T R A C T

We derive two new versions of Cooper's formula for the iterated Askey–Wilson operator. Using the second version of Cooper's formula and the Leibniz rule for the iterated Askey–Wilson operator, we derive several formulas involving this operator. We also give new proofs of Rogers' summation formula for ${}_6\phi_5$ series, Watson's transformation, and we establish a Rodriguez type operational formula for the Askey–Wilson polynomials. In addition we establish two integration by parts formulas for integrals involving the iterated Askey–Wilson operator. Using the first of these integration by parts formulas, we derive a two parameter generating function for the Askey–Wilson polynomials. A generalization of the Leibniz rule for the iterated Askey–Wilson operator is also given and used to derive a multi-sum identity.

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Summation theorem
Transformation

1. Introduction

This work is a contribution to the development of the theory of basic hypergeometric functions through the Askey–Wilson operators and polynomials. This line of research started in [8] where Ismail showed that the q-Pfaff–Saalschütz theorem and the Sears transformation follow from a q-Taylor series expansions for polynomials. This is the approach followed in [9] to treat summation theorems. The Askey–Wilson operator and polynomials were introduced in the seminal work [4]. The analogue of integration by parts for the Askey–Wilson operator was found later in [5]. This analogue gives the adjoint of the Askey–Wilson operator on a certain weighted L_2 space.

Throughout this work we will assume that $q \in (0, 1)$ and by $\mathcal{D}_q$ we will denote the Askey–Wilson operator. Cooper [6] gave an expression for $\mathcal{D}_q^n f$ in terms of dilations of the function f. In Section 2 we will establish two modified versions of Cooper's formula, the second one of which is suitable for deriving summation and transformation formulas for basic hypergeometric series. Then we will consider several applications of our second version of Cooper's formula and the Leibniz rule for the operator $\mathcal{D}_q$ including new derivations of Rogers' summation formula for very-well-poised ${}_6\phi_5$ series, Watson's transformation, and an operational representation for the Askey–Wilson polynomials which resembles the Rodriguez formula for these polynomials.

In Section 3 we will establish two integration by parts formulas for the operator $\mathcal{D}_q^n$. The first integration by parts formula for $\mathcal{D}_q^n$ is quite general and explicit and contains a double sum of residual terms. The case $n = 1$ of this formula generalizes the integration by parts formula for the operator $\mathcal{D}_q$ given in [5] (see also [9]). The second integration by parts formula for $\mathcal{D}_q^n$ is obtained by a simple iteration of the case $n = 1$ of the first integration by parts formula for $\mathcal{D}_q^n$.

In Section 4 we will use the first integration parts formula for the operator $\mathcal{D}_q^n$ from Section 3 to derive a two parameter generating function for the Askey–Wilson polynomials. We will discuss two applications of this result: a symmetry property of a certain ${}_4\phi_3$ series and a summation formula for an ${}_8\phi_7$ series.

In the last Section 5 we will establish a generalized Leibniz rule for $\mathcal{D}_q^n$ acting on a product of functions and we will use this rule to derive an identity reducing a multi-sum to a very-well-poised series.

There is a rich literature on expansions in polynomials including the Askey–Wilson polynomials. We refer the interested reader to the works [3,10–12,14,16–18].

In the rest of the introduction we recall some basic definitions and properties from the theory basic hypergeometric series or q-series as in [2,7,9]. We assume the reader is familiar with the standard notions of q-series as in [2,7,9].

The Askey–Wilson polynomials $\{p_n(x; \mathbf{a}|q)\}$ are defined by [2,4,7,9]