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Quadratic forms and congruences for ℓ-regular partitions modulo 3, 5 and 7

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ABSTRACT

Let $b_\ell(n)$ be the number of ℓ-regular partitions of n. We show that the generating functions of $b_\ell(n)$ with $\ell = 3, 5, 6, 7$ and 10 are congruent to the products of two items of Ramanujan's theta functions $\psi(q)$, $f(-q)$ and $(q;q)^3_\infty$ modulo 3, 5 and 7. So we can express these generating functions as double summations in q. Based on the properties of binary quadratic forms, we obtain vanishing properties of the coefficients of these series. This leads to several infinite families of congruences for $b_\ell(n)$ modulo 3, 5 and 7.

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1. Introduction

An ℓ-regular partition of n is a partition of n such that none of its parts is divisible by ℓ. Denote the number of ℓ-regular partitions of n by $b_\ell(n)$. The arithmetic properties, the divisibility and the distribution of $b_\ell(n)$ have been widely studied in recent years.

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Alladi [1] studied the 2-adic behavior of $b_2(n)$ and $b_4(n)$ from a combinatorial point of view and obtained the divisibility results for small powers of 2. Lovejoy [12] proved the divisibility and distribution properties of $b_2(n)$ modulo primes $p \geq 5$ by using the theory of modular forms. Gordon and Ono [7] proved the divisibility properties of $b_\ell(n)$ modulo powers of the prime divisors of ℓ. Later, Ono and Penniston [14] studied the 2-adic behavior of $b_2(n)$. And Penniston [15] derived the behavior of p^a-regular partitions modulo p^j using the theory of modular forms.

The arithmetic properties of $b_\ell(n)$ modulo 2 have been widely investigated. Andrews, Hirschhorn and Sellers [2] derived some infinite families of congruences for $b_4(n)$ modulo 2. By applying the 2-dissection of the generating function of $b_5(n)$, Hirschhorn and Sellers [8] obtained many Ramanujan-type congruences for $b_5(n)$ modulo 2. Xia and Yao [16] established several infinite families of congruences for $b_9(n)$ modulo 2. Cui and Gu [4] derived congruences for $b_\ell(n)$ modulo 2 with $\ell = 2, 4, 5, 8, 13, 16$ by employing the p-dissection formulas of Ramanujan's theta functions $\psi(q)$ and $f(-q)$.

As for the arithmetic properties of $b_\ell(n)$ modulo 3, Cui and Gu [5] and Keith [9] and Xia and Yao [17] studied respectively the congruences for $b_9(n)$ modulo 3. Lin and Wang [11] showed that 9-regular partitions and 3-cores satisfy the same congruences modulo 3 and further generalized Keith's conjecture and derived a stronger result. Furcy and Penniston [6] obtained congruences for $b_\ell(n)$ modulo 3 with $\ell = 4, 7, 13, 19, 25, 34, 37, 43, 49$ by using the theory of modular forms.

Notice that all the above congruences for $b_\ell(n)$ were proven by using modular forms or elementary q-series manipulations. In this paper, we take a different approach which is based on the properties of binary quadratic forms. Lovejoy and Osburn [13] generalized the congruences modulo 3 for four types of partitions by employing the representations of numbers as certain quadratic forms. Employing the arithmetic properties of quadratic forms, Kim [10] proved that the number of overpartition pairs of n is almost always divisible by 2^8.

We derive infinite families of congruence relations for ℓ-regular partitions with $\ell = 3, 5, 6, 7, 10$ modulo 3, 5 and 7 by establishing a general method (see Proposition 2.1). Our method is based on a bivariate extension of Cui and Gu's approach [4].

Notice that the generating function of $b_\ell(n)$ is given by

$$B_\ell(q) = \sum_{n=0}^{\infty} b_\ell(n) q^n = \frac{(q^\ell; q^\ell)_\infty}{(q; q)_\infty},$$

where

$$(q; q)_\infty = \prod_{i=1}^{\infty} (1 - q^i)$$

is the standard notation in q-series. Let $\psi(q)$ and $f(-q)$ be Ramanujan's theta functions given by

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