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Characterizing binary matroids with no P_9-minor

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ABSTRACT

In this paper, we give a complete characterization of binary matroids with no P_9-minor. A 3-connected binary matroid M has no P_9-minor if and only if M is a 3-connected regular matroid, a binary spike with rank at least four, one of the internally 4-connected non-regular minors of a special 16-element matroid Y_{16}, or a matroid obtained by 3-summing copies of the Fano matroid to a 3-connected cographic matroid $M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$ $(n \geq 2)$. Here the simple graphs $K'_{3,n}$, $K''_{3,n}$, and $K'''_{3,n}$ are obtained from $K_{3,n}$ by adding one, two, or three edges in the color class of size three, respectively.

1. Introduction

It is well known that the class of binary matroids consists of all matroids without any $U_{2,4}$-minor, and the class of regular matroids consists of matroids without any $U_{2,4}$, F_7, or F_7^*-minor. Kuratowski's Theorem states that a graph is planar if and only if it has no minor that is isomorphic to $K_{3,3}$ or K_5. These examples show that characterizing a class of graphs or matroids without certain minors is often of fundamental importance. We say that a matroid is N-*free* if it does not contain a minor that is isomorphic to N.

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A 3-connected matroid M is said to be internally 4-connected if for any 3-separation of M, one side of the separation is either a triangle or a triad.

There is much interest in characterizing binary matroids without small 3-connected minors. For any 3-connected matroid N, since non-3-connected N-free matroids are precisely those that are constructed from 3-connected N-free matroids using 1- and 2-sum operations, in order to determine all N-free matroids, one only needs to determine all 3-connected N-free matroids. There is only one 3-connected binary matroid with six elements, namely, W_3, where W_n denotes both the wheel graph with n-spokes and the cycle matroid of W_n. There are exactly two 7-element 3-connected binary matroids, F_7 and F_7^*. There are three 8-element 3-connected binary matroids, W_4, S_8, and $AG(3, 2)$, and there are eight 9-element 3-connected binary matroids: $M(K_{3,3})$, $M^*(K_{3,3})$, Prism, $M(K_5\backslash e)$, P_9, P_9^*, binary spike Z_4 and its dual Z_4^*.

$\vert E(M)\vert$	3-connected binary matroids
6	W_3
7	F_7, F_7^*
8	W_4, S_8, $AG(3, 2)$
9	$M(K_{3,3})$, $M^*(K_{3,3})$, $M(K_5\backslash e)$, *Prism*, P_9, P_9^*, Z_4, Z_4^*

For each matroid N in the above list with fewer than nine elements, with the exception of $AG(3, 2)$, the problem of characterizing 3-connected binary N-free matroids is completely solved. Since every 3-connected binary matroid having at least four elements has a W_3-minor, the class of 3-connected binary matroids excluding W_3 contains only the trivial 3-connected matroids with at most three elements. Seymour in [12] determined all 3-connected binary matroids with no F_7-minor (F_7^*-minor). Any such matroid is either regular or is isomorphic to F_7^* (F_7). In [9], Oxley characterized all 3-connected binary W_4-free matroids. These are exactly $M(K_4)$, F_7, F_7^*, binary spikes Z_r, Z_r^*, $Z_r\backslash t$, $Z_r\backslash y_r$ ($r \geq 4$), and the trivial 3-connected matroids with at most three elements. It is an easy corollary of Seymour's Splitter Theorem that F_7, F_7^*, and $AG(3, 2)$ are the only 3-connected binary non-regular matroids without any S_8-minor. At this point, not much is known about $AG(3, 2)$-free matroids.

For each 9-element matroid on the above list there are some partial results. In an AMS Memoir [8], Mayhew, Royle, and Whittle characterized all internally 4-connected binary $M(K_{3,3})$-free matroids. Mayhew and Royle [7], and independently Kingan and Lemos [5], determined all internally 4-connected binary Prism-free (therefore, $M(K_5\backslash e)$-free) matroids. These results are not complete characterizations of binary N-free matroids for the corresponding N because the 3-connected binary N-free matroids are yet to be determined. Since Z_4 has an $AG(3, 2)$-minor, characterizing binary Z_4-free matroids is an even harder problem. Oxley [9] determined all 3-connected binary matroids that contain neither a P_9- nor P_9^*-minor.

Theorem 1.1. *Let M be a binary matroid. Then M is 3-connected having no minor isomorphic to P_9 or P_9^* if and only if*