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Multi-cores, posets, and lattice paths [☆]

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ABSTRACT

Hooks are prominent in representation theory (of symmetric groups) and they play a role in number theory (via cranks associated to Ramanujan's congruences). A partition of a positive integer n has a Young diagram representation. To each cell in the diagram there is an associated statistic called hook length, and if a number t is absent from the diagram then the partition is called a t-core. A partition is an (s, t)-core if it is both an s- and a t-core. Since the work of Anderson on (s, t)-cores, the topic has received growing attention. This paper expands the discussion to multiple-cores. More precisely, we explore $(s, s + 1, \dots, s + k)$-core partitions much in the spirit of a recent paper by Stanley and Zanello. In fact, our results exploit connections between three combinatorial objects: multi-cores, posets and lattice paths (with a novel generalization of Dyck paths). Additional results and conjectures are scattered throughout the paper. For example, one of these statements implies a curious symmetry for twin-coprime $(s, s + 2)$-core partitions.

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0. Introduction

Let S be any set of positive integers. Say that a is *generated* by S if a can be written as a non-negative linear combination of the elements of S. Following the notation of [9], we define P_S to be the set whose elements are positive integers not generated by S. Equivalently, $k \in P_S$ if $\alpha_k = 0$ in the generating function given by

$$\prod_{s \in S} \frac{1}{1 - x^s} = \sum_{k \geq 0} \alpha_k \, x^k. \tag{0.1}$$

This is reminiscent of the *Frobenius coin exchange problem*. We make P_S into a poset by defining the cover relation so that a *covers* b (written $a \gtrdot b$) if and only if $a - b \in S$. For example, see Fig. 1. Note that P_S is finite if and only if the elements of S are relatively prime (no $d > 1$ divides every $s \in S$).

We depict a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ by its French Ferrers diagram. The hook length of a cell c in the diagram of λ is the number of cells directly north or east of c including itself. It is denoted by $\text{hook}_\lambda(c)$, or just $\text{hook}(c)$ when the partition is clear. For any positive integer s, we say λ an s-core if its diagram contains no cell c so that s divides $\text{hook}(c)$. For example, see Fig. 2.

Let $(P, <_P)$ be a poset. We say that a set $I \subseteq P$ is a *lower ideal* of this poset if $a <_P b$ and $b \in I$ implies $a \in I$. The work of [4] gives a natural bijection between s-cores and lower ideals of $P_{\{s\}}$. In particular, this bijection associates the lower ideal I of $P_{\{s\}}$ with the s-core whose first column has hook lengths given by I. For example, the 4-core in Fig. 2 corresponds to the lower ideal $\{1, 2, 5, 9\}$ of $P_{\{4\}}$.

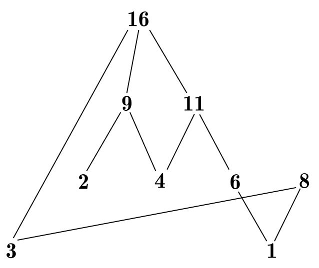

Fig. 1. The poset $P_{\{5,7,13\}}$.

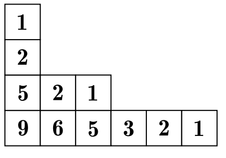

Fig. 2. The French Ferrers diagram of the 4-core $(6, 3, 1, 1)$ with hook lengths marked.