Advances in Applied Mathematics 71 (2015) 125–137

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

A new companion to Capparelli's identities

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ARTICLE INFO

Article history:
Received 12 June 2015
Accepted 9 September 2015
Available online 18 September 2015

MSC:
05A15
05A17
05A20
11B34
11B37
11P83

Keywords:
Capparelli's identity
Integer partitions
q-binomial identities
Partition identities
Partition inequalities

ABSTRACT

We discuss a new companion to Capparelli's identities. Capparelli's identities for $m = 1, 2$ state that the number of partitions of n into distinct parts not congruent to m, $-m$ modulo 6 is equal to the number of partitions of n into distinct parts not equal to m, where the difference between parts is greater than or equal to 4, unless consecutive parts are either both consecutive multiples of 3 or add up to a multiple of 6. In this paper we show that the set of partitions of n into distinct parts where the odd-indexed parts are not congruent to m modulo 3, the even-indexed parts are not congruent to $3 - m$ modulo 3, and $3l + 1$ and $3l + 2$ do not appear together as consecutive parts for any integer l has the same number of elements as the above mentioned Capparelli's partitions of n. In this study we also extend the work of Alladi, Andrews and Gordon by providing a complete set of generating functions for the refined Capparelli partitions, and conjecture some combinatorial inequalities.

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http://dx.doi.org/10.1016/j.aam.2015.09.012

1. Introduction and notation

A partition π is a finite, non-increasing sequence of positive integers $(\pi_1, \pi_2, \ldots, \pi_k)$. The π_i are called parts of the partition π, and $\pi_1 + \pi_2 + \cdots + \pi_k$ is called the norm of π. We call π "a partition of n" if the norm of π is n. Conventionally, we define empty sequence as the only partition of 0. Throughout this paper we assume that a and q are complex numbers where $|q| < 1$, L is a positive integer, and $m \in \{1, 2\}$. We use the standard notations as in [3] and [10]:

$$(a)_L := (a;q)_L = \prod_{n=0}^{L-1}(1 - aq^n),$$

$$(a_1, a_2, \ldots, a_k; q)_L = (a_1;q)_L(a_2;q)_L \ldots (a_k;q)_L,$$

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n).$$

We define the q-binomial and q-trinomial coefficients, respectively, as

$$\begin{bmatrix} k \\ n \end{bmatrix}_q := \begin{cases} \frac{(q)_k}{(q)_n (q)_{k-n}} & \text{for } k \geq n \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\begin{bmatrix} k \\ n, r \end{bmatrix}_q := \begin{bmatrix} k \\ n \end{bmatrix}_q \begin{bmatrix} k-n \\ r \end{bmatrix}_q = \begin{cases} \frac{(q)_k}{(q)_n (q)_r (q)_{k-n-r}} & \text{for } k \geq n + r \geq n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $C_m(n)$ be the number of partitions of n into distinct parts where no part is congruent to $\pm m$ modulo 6. Define $D_m(n)$ to be the number of partitions of n into parts, not equal to m, where the minimal difference between consecutive parts is 2, in fact, the difference between consecutive parts is greater than or equal to 4 unless these parts either are both consecutive multiples of 3 (yielding a difference of 3) or add up to a multiple of 6 (possibly giving a difference of 2).

In 1988, S. Capparelli stated two conjectures for C_m and D_m in his thesis [7]. The first one was later proven by G.E. Andrews [2] in 1992 during the Centenary Conference in Honor of Hans Rademacher. Two years later Lie's theoretic proof was supplied by Tamba and Xie [12] and by Capparelli [8]. The first of Capparelli's conjectures was stated and proven in the form of Theorem 1.1.

Theorem 1.1 *(Andrews 1992). For any non-negative integer n,*

$$C_1(n) = D_1(n).$$