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On the ideal of orthogonal representations of a graph in $\mathbb{R}^2$

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In this paper, we study orthogonal representations of simple graphs G in $\mathbb{R}^d$ from an algebraic perspective in case $d = 2$. Orthogonal representations of graphs, introduced by Lovász, are maps from the vertex set to $\mathbb{R}^d$ where non-adjacent vertices are sent to orthogonal vectors. We exhibit algebraic properties of the ideal generated by the equations expressing this condition and deduce geometric properties of the variety of orthogonal embeddings for $d = 2$ and $\mathbb{R}$ replaced by an arbitrary field. In particular, we classify when the ideal is radical and provide a reduced primary decomposition if $\sqrt{-1} \notin K$. This leads to a description of the variety of orthogonal embeddings as a union of varieties defined by prime ideals. In particular, this applies to the motivating case $K = \mathbb{R}$.

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1. Introduction

Orthogonal representations of graphs were introduced by Lovász in 1979 [9]. In [9] and subsequent work it has been shown that they are intimately related to important combinatorial properties of graphs (see [11, Ch. 9]). More precisely, let G be a finite simple graph on vertex set $V(G) = [n] := \{1, \dots, n\}$ and edge set $E(G) \subseteq \binom{[n]}{2}$ and let $d \geq 1$ be an integer. By $\overline{G}$ we denote the complementary graph of G with edge set $E(\overline{G}) = \binom{[n]}{2} \setminus E(G)$. An orthogonal representation of G in $\mathbb{R}^d$ is a map φ from $[n]$ to $\mathbb{R}^d$ such that for any edge $\{i, j\} \in E(\overline{G})$ in the complementary graph, the vectors $\varphi(i)$ and $\varphi(j)$ are orthogonal with respect to the standard scalar product in $\mathbb{R}^d$. Formulated differently, if we identify the image of the vertex i with the i-th row $(u_{i1}, \dots, u_{id})$ of an $(n \times d)$-matrix $U = (u_{ij})_{(i,j)\in[n]\times[d]} \in \mathbb{R}^{n\times d}$, then the set of all orthogonal representations of the graph G is the vanishing set in $\mathbb{R}^{n\times d}$ of the ideal $L_{\overline{G}} \subset \mathbb{R}[x_{ij} \;\; i = 1, \dots, n, \; j = 1, \dots, d]$, where $L_{\overline{G}}$ is generated by the homogeneous polynomials

$$\sum_{k=1}^{d} x_{ik}x_{jk} \tag{1}$$

for $\{i, j\} \in E(\overline{G})$. We write $\mathrm{OR}_d^{\mathbb{R}}(G) \subseteq \mathbb{R}^{n\times d}$ for the variety of orthogonal representations of G.

The first study of $L_{\overline{G}}$ and the geometry of $\mathrm{OR}_d^{\mathbb{R}}(G)$ can be found in [10]. For that reason we would like to call the ideal $L_{\overline{G}}$ of orthogonal graph representations of G the *Lovász–Saks–Schrijver ideal* of G. Clearly, the variety $\mathrm{OR}_d^{\mathbb{R}}(G)$ contains many degenerate representations where, for example, one of the vertices is represented by the zero vector. To avoid this kind of degeneracy, Lovász, Saks and Schrijver in [10] consider general-position orthogonal representations, that is, orthogonal representations in which any d representing vectors are linearly independent. In [10, Thm. 1.1] they prove the remarkable fact that G has such a representation in $\mathbb{R}^d$ if and only if G is $(n-d)$-connected in which case $L_{\overline{G}}$ is a prime ideal.

While the results from [10] express properties of $\mathrm{OR}_d^{\mathbb{R}}(G)$ and $L_{\overline{G}}$ in terms of graph-theoretic properties of G, it is sometimes convenient to interchange the role of G and $\overline{G}$. Therefore, in our subsequent work we will refer to L_G or $L_{\overline{G}}$ depending on which point of view is more suitable for the given context.

In this paper we want to understand some algebraic properties of $L_{\overline{G}}$ and the geometry of the variety $\mathrm{OR}_d^{K}(G)$ of orthogonal representations of G for general G and over an arbitrary field K. This appears to be a hard task and we confine ourselves to the first interesting case, when $d = 2$ and K an arbitrary field. Note that, for $d = 1$ the ideal L_G is a monomial ideal known as the edge ideal of G and a well studied object (see for example [5, Ch. 9] or [13]).

For an easier reading in the case $d = 2$ we rename the two variables x_{i1}, x_{i2} corresponding to the coordinates of the i-th vertex as x_i, y_i and consider L_G as an ideal in