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Double Aztec rectangles

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ABSTRACT

We investigate the connection between lozenge tilings and domino tilings by introducing a new family of regions obtained by attaching two different Aztec rectangles. We prove a simple product formula for the generating functions of the tilings of the new regions, which involves the statistics as in the Aztec diamond theorem (Elkies et al. (1992) [2,3]). Moreover, we consider the connection between the generating function and MacMahon's q-enumeration of plane partitions fitting in a given box

1. Introduction

A *plane partition* is a rectangular array of non-negative integers so that all columns are weakly decreasing from top to bottom and all rows are weakly decreasing from left to right. A plane partition having a rows and b columns with entries at most c is identified with its 3-D interpretation — a stack of unit cubes fitting in an $a \times b \times c$ box. The

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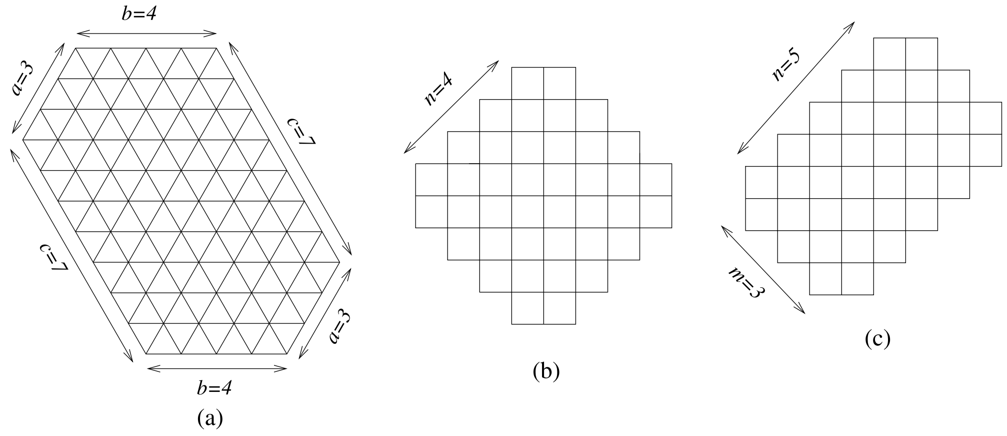

Fig. 1.1. (a) The hexagon $H_{3,4,7}$. (b) The Aztec diamond $\mathcal{AD}_4$. (c) The Aztec rectangle $\mathcal{AR}_{4,6}$.

latter stack in turn corresponds to a lozenge tiling of a centrally symmetric hexagon of side-lengths a, b, c, a, b, c (in clockwise order, starting from the northwest side) on the triangular lattice. We denote this hexagon by $H_{a,b,c}$ (see Fig. 1.1(a) for an example). Here, a *lozenge* (or *unit rhombus*) is union of any two unit equilateral triangles sharing an edge; and a *lozenge tiling* of a region (on the triangular lattice) is a covering of the region by lozenges so that there are no gaps or overlaps.

Let q be an indeterminate. The *q-integer* $[n]_q$ is defined as $[n]_q := 1 + q + \ldots + q^{n-1}$. MacMahon [8] proved that

$$\sum_{\pi} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{[i+j+k-1]_q}{[i+j+t-2]_q}, \tag{1.1}$$

where the sum is taken over all plane partitions π fitting in an $a \times b \times c$ box and where $|\pi|$ is the number of unit cubes in π (i.e. the *volume* of π). By letting $q = 1$, this deduces that

$$\mathbb{T}\big(H_{a,b,c}\big) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{t=1}^{c} \frac{i+j+t-1}{i+j+t-2}, \tag{1.2}$$

where we denote $\mathbb{T}(R)$ for the number of tilings of a region R.

We now consider regions on a different lattice, the square lattice. On this lattice, we are interested in *domino tilings*–coverings of a region by dominoes so that there are no gaps or overlaps. Here, a *domino* is union of any two unit squares sharing an edge. Two central objects in enumeration of domino tilings are the Aztec diamond (see Fig. 1.1(b) for the Aztec diamond of order 4) and its natural generalization, the Aztec rectangle (see Fig. 1.1(c) for an Aztec rectangle of order 3×5). We denote by $\mathcal{AD}_n$ the Aztec diamond of order n, and by $\mathcal{AR}_{m,n}$ the Aztec rectangle of order $m \times n$. One of the crucial results in