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Spanning trees in random series-parallel graphs ☆

Julia Ehrenmüller [a], Juanjo Rué [b,*]

[a] *Technische Universitat Hamburg–Harburg, Institut für Mathematik, Am Schwarzenberg-Campus 3, 21073 Hamburg, Germany*
[b] *Freie Universität Berlin, Institut für Mathematik und Informatik, Arnimallee 3, 14195 Berlin, Germany*

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By means of analytic techniques we show that the expected number of spanning trees in a connected labelled series-parallel graph on n vertices chosen uniformly at random satisfies an estimate of the form

$$s\varrho^{-n}(1+o(1)),$$

where s and ϱ are computable constants, the values of which are approximately $s \approx 0.09063$ and $\varrho^{-1} \approx 2.08415$. We obtain analogous results for subfamilies of series-parallel graphs including 2-connected series-parallel graphs, 2-trees, and series-parallel graphs with fixed excess.

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* Corresponding author.
E-mail addresses: julia.ehrenmueller@tuhh.de (J. Ehrenmüller), jrue@zedat.fu-berlin.de (J. Rué).

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1. Introduction

The study of spanning trees and their enumeration is a central question in graph theory and in combinatorial optimisation. It is well known that the number of spanning trees of a given graph G is an evaluation of its Tutte polynomial (see for instance [7]). A lot of research has been devoted to study estimates of this number when dealing with restricted graph families. For instance, various results have been obtained for regular graphs and for graphs with degree constraints (see e.g. [1,29–31]).

The enumeration of graphs with a distinguished spanning tree has also been studied extensively in the context of planar maps (namely, embedded connected planar graphs in the sphere, see Schaeffer's Chapter in [10] for an introduction to this area). The first result in this context was obtained in the sixties by Mullin who was studying the number of rooted planar maps on n edges with a distinguished spanning tree [33]. Mullin determined that this number is C_nC_{n+1}, where C_n stands for the n-th Catalan number. Such formula was explained later by Cori, Dulucq and Viennot by means of Baxter permutations [15], and by Bernardi using a direct bijection with pairs of plane trees with n and $n+1$ edges, respectively (see [3]). Recently, Bousquet-Mélou and Courtiel investigated the enumeration of regular planar maps carrying a distinguished spanning *forest*, as well as the connections of these counting formulae with statistical models as the Potts model [11] (see also [4] for the connection of spanning trees on maps and the Tutte polynomial).

In this paper we study spanning trees in series-parallel graphs. A graph is *series-parallel* (or SP for short) if it is K_4-minor free. Over the past few decades, SP graphs have been extensively studied from various points of view both in graph theory and computer science. In particular, being a subclass of planar graphs and a superclass of outerplanar graphs, SP graphs turned out to serve well as a pre-stage for the analysis and study of problems on planar graphs. Indeed, the family of SP graphs is the prototype of the so-called *subcritical* graph class family (see e.g. [21,25]). In a typical connected graph in such a family, maximal 2-connected subgraphs (also called *blocks*) are small compared with the total size of the graph. This behaviour arises as a consequence of a subcritical composition phenomenon which appears in the specification of the generating functions associated with connected graphs of the family.

In this paper we focus on enumerative problems defined on SP graphs. To this purpose, let us quickly state the following alternative definition of SP graphs that gives more insight into their structure and also justifies their name. Let G be a graph and let s and t be two of its vertices. We say G is *series-parallel with terminals s and t* if G can be turned into the single edge $\{s, t\}$ by a sequence of the following operations: replacement of a pair of *parallel edges* (i.e. edges that have two common endpoints) by a single edge, or replacement of a pair of *series edges* (i.e. non-parallel edges that have a common endpoint of degree 2) by a single edge. A graph G is *2-terminal series-parallel* if there are vertices s and t in G such that G is series-parallel with terminals s and t. Finally, a