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Asymptotic shape of the convex hull of isotropic log-concave random vectors

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ABSTRACT

Let $x_1, \ldots, x_N$ be independent random points distributed according to an isotropic log-concave measure μ on $\mathbb{R}^n$, and consider the random polytope

$$K_N := \mathrm{conv}\{\pm x_1, \ldots, \pm x_N\}.$$

We provide sharp estimates for the quermaßintegrals and other geometric parameters of K_N in the range $cn \leqslant N \leqslant \exp(n)$; these complement previous results from [13] and [14] that were given for the range $cn \leqslant N \leqslant \exp(\sqrt{n})$. One of the basic new ingredients in our work is a recent result of E. Milman that determines the mean width of the centroid body $Z_q(\mu)$ of μ for all $1 \leqslant q \leqslant n$.

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1. Introduction

The purpose of this work is to add new information on the asymptotic shape of random polytopes whose vertices have a log-concave distribution. Without loss of generality we shall assume that this distribution is also isotropic. Recall that a convex body K in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered, i.e. its center of mass is at the origin, and its inertia matrix is a multiple of the identity: there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2 \tag{1.1}$$

for every θ in the Euclidean unit sphere S^{n-1}. More generally, a log-concave probability measure μ on $\mathbb{R}^n$ is called isotropic if its center of mass is at the origin and its inertia matrix is the identity; in this case, the isotropic constant of μ is defined as

$$L_\mu := \sup_{x\in\mathbb{R}^n} \left(f_\mu(x)\right)^{1/n}, \tag{1.2}$$

where f_μ is the density of μ with respect to the Lebesgue measure. Note that a centered convex body K of volume 1 in $\mathbb{R}^n$ is isotropic if and only if the log-concave probability measure μ_K with density $x \mapsto L_K^n \mathbf{1}_{K/L_K}(x)$ is isotropic.

A very well-known open question in the theory of isotropic measures is the hyperplane conjecture, which asks if there exists an absolute constant $C > 0$ such that

$$L_n := \sup\{L_\mu : \mu \text{ is an isotropic log-concave measure on } \mathbb{R}^n\} \leqslant C \tag{1.3}$$

for all $n \geqslant 1$. Bourgain proved in [9] that $L_n \leqslant c\sqrt[4]{n}\log n$ (more precisely, he showed that $L_K \leqslant c\sqrt[4]{n}\log n$ for every isotropic symmetric convex body K in $\mathbb{R}^n$), while Klartag [18] obtained the bound $L_n \leqslant c\sqrt[4]{n}$. A second proof of Klartag's estimate appears in [20].

The study of the asymptotic shape of random polytopes whose vertices have a log-concave distribution was initiated in [13] and [14]. Given an isotropic log-concave measure μ on $\mathbb{R}^n$, for every $N \geqslant n$ we consider N independent random points $x_1, \dots, x_N$ distributed according to μ and define the random polytope $K_N := \mathrm{conv}\{\pm x_1, \dots, \pm x_N\}$. The main idea in these works was to compare K_N with the L_q-centroid body of μ for a suitable value of q; roughly speaking, K_N is close to the body $Z_{\log(2N/n)}(\mu)$ with high probability. Recall that the L_q-centroid bodies $Z_q(\mu)$, $q \geqslant 1$, are defined through their support function $h_{Z_q(\mu)}$, which is given by

$$h_{Z_q(\mu)}(y) := \|\langle \cdot, y\rangle\|_{L_q(\mu)} = \left(\int_{\mathbb{R}^n} |\langle x, y\rangle|^q d\mu(x)\right)^{1/q}. \tag{1.4}$$

These bodies incorporate information about the distribution of linear functionals with respect to μ. The L_q-centroid bodies were introduced, under a different normalization, by Lutwak and Zhang in [24], while in [29] for the first time, and in [30] later on, Paouris