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On a surface formed by randomly gluing together polygonal discs

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Starting with a collection of n oriented polygonal discs, with an even number N of sides in total, we generate a random oriented surface by randomly matching the sides of discs and properly gluing them together. Encoding the surface by a random permutation γ of $[N]$, we use the Fourier transform on S_N to show that γ is asymptotic to the permutation distributed uniformly on the alternating group A_N (A_N^c resp.) if $N-n$ and $N/2$ are of the same (opposite resp.) parity. We use this to prove a local central limit theorem for the number of vertices on the surface, whence also for its Euler characteristic χ. We also show that with high probability (as $N\to\infty$, uniformly in n) the random surface consists of a single component, and thus has a well-defined genus $g=1-\chi/2$, which is asymptotic to a Gaussian random variable, with mean $(N/2-n-\log N)/2$ and variance $(\log N)/4$.

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1. Introduction and main results

In this paper we study random surfaces obtained by gluing, uniformly at random, sides of n polygons with various (not necessarily equal) numbers of sides. We call this scheme of generating a surface the *map model*. (A model dual to the map model is very important for algebraic geometry [15]. It can be generalized to hypermaps; in [5] it is called the σ-*model*.) In the map model the interiors of polygons represent countries (*faces*); the glued sides represent boundaries between countries (*edges*). Thus the map model can be considered as a graph embedded into the surface such that the faces correspond to the original polygons.

This model generalizes the random map model, motivated by studies in quantum gravity, of Pippenger and Schleich [22], in which all polygons are triangles. For the Euler characteristic χ of the randomly triangulated surface they proved that $\mathrm{E}[\chi] = n/2 - \log n + O(1)$ and $\mathrm{Var}(\chi) = \log n + O(1)$, as well as made startlingly sharp conjectures regarding the remainder terms $O(1)$ based on simulations and results for similar models. The case when the numbers of sides of all polygons are equal, gluings of k-gons ($k \geqslant 3$), was considered by Gamburd in [12]. His breakthrough result was that (for $2\,\mathrm{lcm}(2,k) \mid kn$) the random permutation of polygon sides, that determines the map, was asymptotically uniform on the alternating group A_{kn}. This implies, for instance, that χ was asymptotic, in distribution, to $n/2$ minus $\mathcal{N}(\log n, \log n)$, the Gaussian variable with mean and variance equal to $\log n$. Fleming and Pippenger [9] used Gamburd's result to prove sharp asymptotic formulas for the first four moments of the Euler characteristic χ, in particular confirming the earlier conjectures for $k = 3$ in [22]. Another special case of this model is when there is only one polygon whose sides are glued in pairs. This case is well studied, popular, and important in combinatorics and the theory of moduli spaces of algebraic curves. The classic paper of Harer and Zagier [13] solved the difficult problem of enumerating the resulting surfaces by genus. Their result was used in [4] to determine the limiting genus distribution for the surface chosen uniformly at random from all such surfaces.

Getting back to the n polygons, their sides are glued in pairs. So the total number of sides N of all polygons must be even, and the resulting map will have $N/2$ edges. We also assume that all polygons are oriented and that in each glued pair the edges are oriented opposite-wise. Thus the resulting surface will be oriented.

The map model can be described in terms of permutations. Label the oriented sides e's (edges) of all polygons by numbers from $[N] := \{1, 2, \ldots, N\}$; e_ℓ will denote the edge labeled ℓ. Let n_j be the number of polygons with j sides, j-gons, and let J stand for the set of all possible numbers of sides of our $n = \sum_j n_j$ polygons, so that $\sum_{j \in J} j n_j = N$ and each map will have n faces. We define the permutation α of $[N]$ as follows: $\alpha(e_\ell) = e_k$ if e_k follows immediately after e_ℓ in one of the n oriented polygons. Thus α has n cycles, each cycle consisting of the edges of the corresponding polygon listed according to the polygon (clockwise) orientation. The set of all such α's is the conjugacy class $\mathcal{C}_{\mathbf{n}}$, $\mathbf{n} := \{n_j\}_{j \in J}$, of all permutations of $[N]$ with n_j cycles of length j. A gluing itself is encoded by a permutation β which is a product of transpositions of edges that are glued to each other;