Advances in Applied Mathematics 73 (2016) 59–98

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Two operators on sandpile configurations, the sandpile model on the complete bipartite graph, and a Cyclic Lemma

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A R T I C L E I N F O

Article history:
Received 2 March 2015
Received in revised form 24 September 2015
Accepted 24 September 2015
Available online 16 November 2015

MSC:
05C57
05A19
91A43

Keywords:
Sandpile model
Complete graphs
Complete bipartite graph
Cyclic lemma

A B S T R A C T

We introduce two operators on stable configurations of the sandpile model that provide an algorithmic bijection between recurrent and parking configurations. This bijection preserves their equivalence classes with respect to the sandpile group. The study of these operators in the special case of the complete bipartite graph $K_{m,n}$ naturally leads to a generalization of the well-known Cyclic Lemma of Dvoretsky and Motzkin, via pairs of periodic bi-infinite paths in the plane having slightly different slopes. We achieve our results by interpreting the action of these operators as an action on a point in the grid $\mathbb{Z}^2$ which is pointed to by one of these pairs of paths. Our Cyclic Lemma allows us to enumerate several classes of polyominoes, and therefore builds on the work of Irving and Rattan (2009), Chapman et al. (2009), and Bonin et al. (2003).

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[1] J.-C. Aval is supported by ANR – PSYCO project (ANR-11-JS02-001).

http://dx.doi.org/10.1016/j.aam.2015.09.018

1. Introduction

The abelian sandpile model is a cellular automaton on a graph. It was the first example of a dynamical system exhibiting a fascinating property called *self-organized criticality*; see [3]. This model has since proved to be a fertile ground from which many new and unlikely results have emerged. One popular example is the correspondence between recurrent configurations of the sandpile model on a graph and spanning trees of the same graph; see e.g. [9].

In the abelian sandpile model on an undirected connected loop-free graph, states are vectors which indicate the number of grains present at every vertex of the graph. A vertex may be toppled when the number of grains at that vertex is not less than the degree of that vertex. When a vertex is toppled, one grain of sand is sent along each incident edge to neighboring vertices. A *sink* is a distinguished vertex in the graph. A *configuration* is an assignment of grains to graph vertices, and a configuration is called *stable* if the number of grains at each vertex other than the sink is less than the degree of that vertex.

Two configurations are called *toppling equivalent* if there is a sequence of topplings of one of the configurations that results in the other. Given a configuration, the configurations that can be obtained from it by any finite sequence of topplings form the toppling equivalence class of this configuration. We study in particular the partition of stable configurations into toppling equivalence classes.

In Section 2 of this paper we consider two operators, ψ and φ, on stable sandpile configurations. These operators are, in a sense, dual to one another. We prove that the fixed points of the operator ψ are the recurrent sandpile configurations and the fixed points of φ are the G-parking sandpile configurations (an extension of the classical parking function to an arbitrary directed graph G). The motivation in introducing the operators ψ and φ was to produce an algorithm that allows one to go from recurrent configurations to G-parking configurations, and vice versa, within the same toppling equivalence class. As a byproduct, we get two dual definitions of recurrent and G-parking configurations.

In Section 3, we consider pairs of periodic bi-infinite paths in the plane defined by a pair of binary words. These binary words describe their respective minimal periods. The two periods differ slightly since one period describes a lattice path from the origin to (m, n) while the other describes a path from the origin to $(m - 1, n)$. For both of these finite paths, amend and prepend that same path to itself an infinite number of times to produce a pair of periodic paths with slightly different periods. We prove a result that we call the Cyclic Lemma (Lemma 3.1) which consists of two parts. The first part shows that the pairs of binary words can be partitioned into sets that each contains m