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Decomposing tensors into frames

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A symmetric tensor of small rank decomposes into a configuration of only few vectors. We study the variety of tensors for which this configuration is a unit norm tight frame.

1. Introduction

A fundamental problem in computational algebraic geometry, with a wide range of applications, is the low rank decomposition of symmetric tensors; see e.g. [1,3,10,20,21]. If $T = (t_{i_1 i_2 \cdots i_d})$ is a symmetric tensor in $\mathrm{Sym}_d(\mathbb{C}^n)$, then such a decomposition is

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$$T = \sum_{j=1}^{r} \lambda_j \mathbf{v}_j^{\otimes d}. \tag{1}$$

Here $\lambda_j \in \mathbb{C}$ and $\mathbf{v}_j = (v_{1j}, v_{2j}, \dots, v_{nj}) \in \mathbb{C}^n$ for $j = 1, 2, \dots, r$. The smallest r for which a representation (1) exists is the *rank* of T. In particular, each $\mathbf{v}_j^{\otimes d}$ is a tensor of rank 1.

An equivalent way to represent a symmetric tensor T is as a homogeneous polynomial

$$T = \sum_{i_1,\dots,i_d=1}^{n} t_{i_1 i_2 \cdots i_d} \cdot x_{i_1} x_{i_2} \cdots x_{i_d}. \tag{2}$$

If $d = 2$, then (2) is the identification of symmetric matrices with quadratic forms. Written as a polynomial, the right hand side of (1) is a linear combination of powers of linear forms:

$$T = \sum_{j=1}^{r} \lambda_j (v_{1j}x_1 + v_{2j}x_2 + \cdots + v_{nj}x_n)^d. \tag{3}$$

The decomposition in (1) and (3) is called *Waring decomposition*. When $d = 2$, it corresponds to orthogonal diagonalization of symmetric matrices. We could subsume the constants λ_i into the vectors $\mathbf{v}_i$ but we prefer to leave (1) and (3) as is, for reasons to be seen shortly. The (projective) variety of all such symmetric tensors is the *r-th secant variety of the Veronese variety*. The vast literature on the geometry and equations of this variety (cf. [18]) forms the mathematical foundation for low rank decomposition algorithms for symmetric tensors.

In many situations one places further restrictions on the summands in (1) and (3), such as being real and nonnegative. Applications to machine learning in [1] concern the case when $r = n$ and the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ form an orthonormal basis of $\mathbb{R}^n$. The article [21] characterizes the *odeco variety* of all tensors that admit such an orthogonal decomposition.

The present paper takes this one step further by connecting tensors to *frame theory* [6,5,9,12,26]. We examine the scenario when the $\mathbf{v}_j$ form a **f**inite **u**nit **n**orm **t**ight **f**rame (or funtf) of $\mathbb{R}^n$, an object of recent interest at the interface of applied functional analysis and algebraic geometry. Consider a configuration $V = (\mathbf{v}_1, \dots, \mathbf{v}_r) \in (\mathbb{R}^n)^r$ of r labeled vectors in $\mathbb{R}^n$. We also regard this as an $n \times r$-matrix $V = (v_{ij})$. We call V a *funtf* if

$$V \cdot V^T = \frac{r}{n} \cdot \mathrm{Id}_n \qquad \text{and} \qquad \sum_{j=1}^{n} v_{ij}^2 = 1 \quad \text{for } i = 1, 2, \dots, r. \tag{4}$$

This is an inhomogeneous system of n^2+r quadratic equations in nr unknowns. The *funtf variety*, denoted $\mathcal{F}_{r,n}$ as in [5], is the subvariety of complex affine space $\mathbb{C}^{n\times r}$ defined by (4). For the state of the art we refer to the article [5] by Cahill, Mixon and Strawn, and the references therein. A detailed review, with some new perspectives, will be given in Section 2.