Advances in Applied Mathematics 69 (2015) 1–45

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Star-cumulants of free unitary Brownian motion

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A R T I C L E I N F O

Article history:
Received 10 September 2014
Received in revised form 7 May 2015
Accepted 7 May 2015
Available online 20 May 2015

MSC:
primary 46L54
secondary 06A07, 60G99

Keywords:
Free probability
Free unitary Brownian motion
Free cumulants
Non-crossing partitions
R-diagonal element
Infinitesimal determining sequence

A B S T R A C T

We study joint free cumulants of u_t and u_t^*, where u_t is a free unitary Brownian motion at time t. We determine explicitly some special families of such cumulants. On the other hand, for a general joint cumulant of u_t and u_t^*, we "calculate the derivative" for $t \to \infty$, when u_t approaches a Haar unitary. In connection to the latter calculation we put into evidence an "infinitesimal determining sequence" which naturally accompanies an arbitrary R-diagonal element in a tracial $*$-probability space.

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[1] Nizar Demni was supported in part by Grant ANR-11-LABEX-0020-01 from ANR, France.
[2] Mathieu Guay-Paquet was supported by a Postdoctoral Fellowship from NSERC, Canada.
[3] Alexandru Nica was supported in part by a Discovery Grant from NSERC, Canada.

http://dx.doi.org/10.1016/j.aam.2015.05.001
0196-8858/

1. Introduction

Let $(u_t)_{t\geq 0}$ be a free unitary Brownian motion in the sense of [3,4] — that is, every u_t is a unitary element in some tracial $*$-probability space $(\mathcal{A}_t, \varphi_t)$, with $\varphi_t(u_t) = e^{-t/2}$, and where the rescaled element $v_t := e^{t/2}u_t$ has S-transform given by

$$S_{v_t}(z) = e^{tz}, \quad z \in \mathbb{C}. \tag{1.1}$$

Closely related to Eq. (1.1), one has a nice formula for the free cumulants of u_t, i.e. for the sequence of numbers $\big(\kappa_n(u_t, \dots, u_t)\big)_{n=1}^{\infty}$, where $\kappa_n : \mathcal{A}_t^n \to \mathbb{C}$ is the n-th free cumulant functional of the space $(\mathcal{A}_t, \varphi_t)$. Indeed, these numbers are the coefficients of the R-transform R_{u_t}. By using the relation between the S-transform and the compositional inverse of the R-transform (which simply says that $zS(z) = R^{\langle -1\rangle}(z)$), one finds that

$$R_{v_t}(z) = \frac{1}{t}W(tz), \quad t > 0, \tag{1.2}$$

where

$$W(y) = y - y^2 + \frac{3}{2}y^3 - \frac{8}{3}y^4 + \cdots + \frac{(-n)^{n-1}}{n!}y^n + \cdots$$

is the Lambert series. Extracting the coefficient of z^n in (1.2) gives the value of $\kappa_n(v_t, \dots, v_t)$, then rescaling back gives

$$\kappa_n(u_t, \dots, u_t) = e^{-nt/2}\kappa_n(v_t, \dots, v_t) = e^{-nt/2}\frac{(-n)^{n-1}}{n!} \cdot t^{n-1}, \; n \in \mathbb{N}, \, t \geq 0. \tag{1.3}$$

In this paper we study joint free cumulants of u_t and u_t^*, that is, quantities of the form

$$\kappa_n\big(\, u_t^{\omega(1)}, \dots, u_t^{\omega(n)}\,\big), \text{ where } n \in \mathbb{N} \text{ and } \omega = (\omega(1), \dots, \omega(n)) \in \{1, *\}^n.$$

The motivation for paying attention to these joint free cumulants comes from looking at the limit $t \to \infty$, when u_t approximates in distribution a Haar unitary. Recall that a unitary u in a $*$-probability space $(\mathcal{A}, \varphi)$ is said to be a Haar unitary when it has the property that $\varphi(u^n) = 0$ for every $n \in \mathbb{Z} \setminus \{0\}$. This property trivially implies $\kappa_n(u, \dots, u) = 0$ for every $n \in \mathbb{N}$, thus the free cumulants of u alone do not look too exciting. However, things become interesting upon considering the larger family of joint free cumulants of u and u^*. There we get the following non-trivial formula, first found in [10]: for $\omega = (\omega(1), \dots, \omega(n)) \in \{1, *\}^n$ one has

$$\kappa_n\big(u^{\omega(1)}, \dots, u^{\omega(n)}\,\big) = \begin{cases} (-1)^{k-1}C_{k-1}, & \text{if } n \text{ is even, } n = 2k, \text{ and} \\ & \quad \omega = (1, *, 1, *, \dots, 1, *) \\ & \quad \text{or } \omega = (*, 1, *, 1, \dots, *, 1), \\ 0, & \text{otherwise,} \end{cases} \tag{1.4}$$