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Combinatorics of hexagonal fully packed loop configurations ☆

Sabine Beil

Universität Wien, Fakultät für Mathematik, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria

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ABSTRACT

This paper introduces fully packed loop configurations of hexagonal shape (HFPLs) as a generalization of triangular fully packed loop configurations. To encode the boundary conditions of an HFPL, a sextuple $(\mathsf{l}_\mathsf{T}, \mathsf{t}, \mathsf{r}_\mathsf{T}; \mathsf{l}_\mathsf{B}, \mathsf{b}, \mathsf{r}_\mathsf{B})$ of 01-words is assigned to it. The first main result of this article establishes necessary conditions for the boundary $(\mathsf{l}_\mathsf{T}, \mathsf{t}, \mathsf{r}_\mathsf{T}; \mathsf{l}_\mathsf{B}, \mathsf{b}, \mathsf{r}_\mathsf{B})$ of an HFPL. The inequality $d(\mathsf{r}_\mathsf{B}) + d(\mathsf{b}) + d(\mathsf{l}_\mathsf{B}) \geq d(\mathsf{l}_\mathsf{T}) + d(\mathsf{t}) + d(\mathsf{r}_\mathsf{T}) + |\mathsf{l}_\mathsf{T}|_1|\mathsf{t}|_0 + |\mathsf{t}|_1|\mathsf{r}_\mathsf{T}|_0 + |\mathsf{r}_\mathsf{B}|_0|\mathsf{l}_\mathsf{B}|_1$ is an example of one such condition (here $|\cdot|_i$ denotes the number of occurrences of i and $d(\cdot)$ denotes the number of inversions). The other main results of this article are expressions in terms of Littlewood–Richardson coefficients for the numbers of HFPLs with boundary $(\mathsf{l}_\mathsf{T}, \mathsf{t}, \mathsf{r}_\mathsf{T}; \mathsf{l}_\mathsf{B}, \mathsf{b}, \mathsf{r}_\mathsf{B})$ such that $d(\mathsf{r}_\mathsf{B}) + d(\mathsf{b}) + d(\mathsf{l}_\mathsf{B}) - d(\mathsf{l}_\mathsf{T}) - d(\mathsf{t}) - d(\mathsf{r}_\mathsf{T}) - |\mathsf{l}_\mathsf{T}|_1|\mathsf{t}|_0 - |\mathsf{t}|_1|\mathsf{r}_\mathsf{T}|_0 - |\mathsf{r}_\mathsf{B}|_0|\mathsf{l}_\mathsf{B}|_1 = 0, 1$.

1. Introduction

This article focuses on the *fully packed loop model* which has its origin in the *six-vertex* model (which is also called *square ice* model) of statistical mechanics. The six-vertex

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E-mail address: sabine.beil@univie.ac.at.

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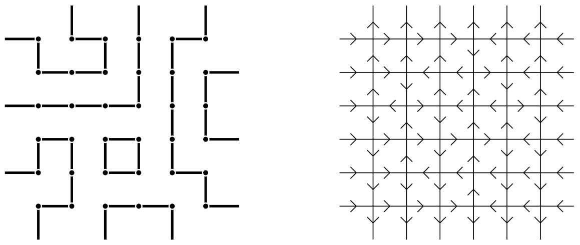
0196-8858/



$$\begin{matrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{matrix}$$

Fig. 1. An FPL, a six-vertex configuration and an ASM; all are of size 6.

model is of specific significance for algebraic combinatorics due to the one-to-one correspondence between six-vertex configurations with *domain wall boundary conditions* (DWBC) on an $n \times n$-square together with $4n$ external edges and *alternating sign matrices* (ASMs) of size n. This correspondence helped to enumerate ASMs of size n, see [8] and [4]. However, in contrast to ASMs and six-vertex configurations, fully packed loop configurations (FPLs), that is, subgraphs of the $n \times n$-square grid together with $2n$ external edges so that each vertex is of degree 2, offer a refined study in dependency on the connectivity of the occupied external edges (these connections are encoded as a *link pattern*). Examples of an FPL, of a six-vertex configuration with DWBC and of an ASM can be found in Fig. 1.

It was conjectured in [10] that the number of FPLs having a fixed link pattern π containing m *nested arches* (denoted π_m) is polynomial in m. In the course of the proof of this conjecture in [1] it was shown that FPLs with link pattern π_m admit a decomposition in which *triangular fully packed loop configurations* (TFPLs) arise. Since then many more nice properties of TFPLs have been discovered, see [5] and [7]. For example, necessary conditions for the *boundary* of a TFPL that is a triple $(u, v; w)$ of 01-words came up in [1], [2] and [5]. One of these conditions states that $d(w) \geq d(u) + d(v)$, where $d(\cdot)$ denotes the number of inversions of a 01-word. This was first proved in [7] for the Dyck-word-case and later in [2] for the general case. For the proof of the general case in [2] it is essential to consider TFPLs together with an orientation of their edges. To be more precise, for *oriented* TFPLs an interpretation of the difference $d(w) - d(u) - d(v)$ in terms of occurrences of certain local configurations is proved. Considering TFPLs with respect to the *excess* $\mathrm{exc}(u, v; w) = d(w) - d(u) - d(v)$ turned out to be fruitful: under the constraint that $\mathrm{exc}(u, v; w) = 0$, it was shown that TFPLs with boundary $(u, v; w)$ are enumerated by the Littlewood–Richardson coefficient $c^{\lambda(w)}_{\lambda(u),\lambda(v)}$, where $\lambda(\cdot)$ denotes the Young diagram associated with a 01-word. First, this was shown for the Dyck-word-case in [6], after which a proof for general case followed (see [2]). Furthermore, this latter paper expressed the number of TFPLs with boundary $(u, v; w)$ such that $\mathrm{exc}(u, v; w) = 1$ in terms of Littlewood–Richardson coefficients.

In this paper fully packed loop configurations of hexagonal shape will be introduced as a generalization of TFPLs. Examples of a TFPL and an HFPL are given in Fig. 2. The goal of this article is to generalize the results for TFPLs mentioned earlier to HFPLs. To that end, each HFPL is assigned a sextuple $(\mathsf{l_T}, \mathsf{t}, \mathsf{r_T}; \mathsf{l_B}, \mathsf{b}, \mathsf{r_B})$ of 01-words that encodes