Advances in Applied Mathematics 67 (2015) 55–74

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

On the shape of a convex body with respect to its second projection body

Christos Saroglou

Department of Mathematics, Texas A&M University, 77840 College Station, TX, USA

A R T I C L E I N F O

Article history:
Received 6 November 2014
Received in revised form 21 March 2015
Accepted 23 March 2015
Available online 11 April 2015

MSC:
primary 52A40
secondary 52A15, 52A20

Keywords:
Projection body
Petty's problem
Reverse isoperimetric inequalities

A B S T R A C T

We prove results relative to the problem of finding sharp bounds for the affine invariant $P(K) = V(\Pi K)/V^{d-1}(K)$. Namely, we prove that if K is a 3-dimensional zonoid of volume 1, then its second projection body $\Pi^2 K$ is contained in $8K$, while if K is any symmetric 3-dimensional convex body of volume 1, then $\Pi^2 K$ contains $6K$. Both inclusions are sharp. Consequences of these results include a stronger version of a reverse isoperimetric inequality for 3-dimensional zonoids established by the author in a previous work, a reduction for the 3-dimensional Petty conjecture to another isoperimetric problem and the best known lower bound up to date for $P(K)$ in 3 dimensions. As byproduct of our methods, we establish an almost optimal lower bound for high-dimensional bodies of revolution.

1. Introduction

Let K be a convex body in $\mathbb{R}^d$, that is convex, compact and with non-empty interior. Denote the volume of K by $V_d(K) = V(K)$ and by h_K its support function. The support function of K is defined as

E-mail addresses: saroglou@math.tamu.edu, christos.saroglou@gmail.com.

http://dx.doi.org/10.1016/j.aam.2015.03.004

$$h_K(x) = \max_{y\in K}\langle x, y\rangle \ , \ x \in \mathbb{R}^d \ .$$

The support function of K is convex and positively homogeneous. Moreover, if L is a different convex body with $L \supsetneq K$, then $h_L \geq h_K$ and $h_K \neq h_L$, therefore K is characterized by its support function. On the other hand, any convex and positively homogeneous function is the support function of a (unique) convex body. The support function is, furthermore, additive with respect to the Minkowski addition: $h_{K+M} = h_K + h_M$, where M is a convex body and $K + M := \{x + y \mid x \in K, \ y \in M\}$.

Define the *projection body* ΠK of K by its support function

$$h_{\Pi K}(u) = V_{d-1}\big(K|u^{\perp}\big) = \frac{1}{2}\int_{S^{d-1}} |\langle u, y\rangle| \ dS_K(y) \ , \ u \in S^{d-1} \ .$$

Here, $K|u^{\perp}$ denotes the orthogonal projection of K on the subspace which is orthogonal to u and $S(K, \cdot)$ stands for the surface area measure of K, viewed as a measure on S^{d-1}. This is defined as

$$S_K(\Omega) = V\Big(\big\{x \in \mathrm{bd}(K) : \exists \ u \in \Omega, \text{ so that } u \text{ is a normal unit vector for } K \text{ at } x\big\}\Big) \ .$$

It is clear that the projection body of a polytope is a *zonotope*, i.e., the Minkowski sum of a finite number of line segments and in general the projection body of a convex body is a *zonoid*, i.e. a limit of zonotopes, in the sense of the Hausdorff metric. Conversely, it can be proven that all zonoids are, up to translation, projection bodies of convex bodies. We refer to [38,8,13,15,14] for more information about support functions and projection bodies.

C.M. Petty [25] proved that ΠK is an affine equivariant, in particular the following expression

$$P(K) := \frac{V(\Pi K)}{V(K)^{d-1}}$$

is invariant under invertible affine maps. Therefore, and since $P(K)$ is continuous with respect to the Hausdorff metric, classical arguments ensure the existence of its extremal values; these are both finite and positive. The determination of these extremals is a difficult and challenging problem in convex and affine geometry. It should be noted that the problem is interesting only for $d \geq 3$. In the plane (see e.g. [33]), the minimizers are all centrally symmetric convex figures and the maximizers are precisely the triangles. Before we state our main results, we would like to say a few words about the history of this problem.

Petty [26] conjectured that

$$P(K) \geq \omega_{d-1}^d \omega_d^{2-d} \ ,$$

with equality if and only if K is an ellipsoid, where ω_d is the volume of the d-dimensional unit ball B_2^d. The Petty conjecture, if true, would be a powerful tool in the study of var-