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Dirac reduction for nonholonomic mechanical systems and semidirect products

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A B S T R A C T

This paper develops the theory of Dirac reduction by symmetry for nonholonomic systems on Lie groups with broken symmetry. The reduction is carried out for the Dirac structures, as well as for the associated Lagrange–Dirac and Hamilton–Dirac dynamical systems. This reduction procedure is accompanied by reduction of the associated variational structures on both Lagrangian and Hamiltonian sides. The reduced dynamical systems obtained are called the implicit Euler–Poincaré–Suslov equations with advected parameters and the implicit Lie–Poisson–Suslov equations with advected parameters. The theory is illustrated with the help of finite and infinite dimensional examples. It is shown that equations of motion for second order Rivlin–Ericksen fluids can be formulated as an infinite dimensional nonholonomic system in the framework of the present paper.

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1. Introduction

Dirac structures in mechanics. Dirac structures are geometric objects that generalize both Poisson brackets and (pre)symplectic structures on manifolds. They were originally developed by Courant, Weinstein [19]; Courant [18] and Dorfman [23] and were named after Dirac's theory of constraints [21].

It turns out that Dirac structures are the appropriate geometric objects in a Hamiltonian setting for the formulation of equations of motion of nonholonomic systems on arbitrary configuration manifolds or, more generally, of implicit Hamiltonian systems appearing as differential-algebraic equations (see, for instance, [54] and [5]). More recently, the notion of an implicit Lagrangian system, which may be a Lagrangian analogue of an implicit Hamiltonian system, was developed by Yoshimura, Marsden [55], where it was shown that nonholonomic mechanical systems and L–C circuits can be formulated as degenerate Lagrangian systems in this context.

For the case of nonholonomic mechanics, given a constraint distribution Δ_Q on the configuration manifold Q, there exists an associated Dirac structure D_{Δ_Q} on the cotangent bundle T^*Q of Q, introduced in [55]. This Dirac structure is not integrable, unless the constraint is holonomic. Given such an induced Dirac structure and a (possibly degenerate) Lagrangian defined on the tangent bundle TQ of Q, a Lagrange–Dirac dynamical system can be defined [55], which provides a geometric formulation of the equations of motion for the nonholonomic mechanical systems. The associated equations of motion are the so-called *implicit Lagrange–d'Alembert equations*. The Lagrange–Dirac system is naturally associated to a variational structure, called the *Lagrange–d'Alembert–Pontryagin principle* whose critical curves are precisely the solutions of the implicit Lagrange–d'Alembert equations.

On the Hamiltonian side, dynamical systems associated to Dirac structures were considered in [22,19,18]. Further applications of Dirac structures to dynamical systems including L–C circuits and nonholonomic systems were developed by van der Schaft, Maschke [54] and Bloch, Crouch [5]. The case of the Dirac structure D_{Δ_Q} induced by a nonholonomic constraint $\Delta_Q \subset TQ$ was considered in [57] to formulate a Hamilton–Dirac system in nonholonomic mechanics.

In the presence of symmetries, there is a well-developed reduction theory for Dirac structures in mechanics for G-invariant systems on Lie groups, both in the unconstrained case and in the constrained (nonintegrable) case [57], where the associated Dirac reduction process (called Lie–Dirac reduction) yields the geometric framework for the study of the Euler–Poincaré–Suslov and Lie–Poisson–Suslov equations. For the more general case of a free and proper Lie group action on an arbitrary configuration manifold, a reduction theory called *Dirac cotangent bundle reduction* was developed for the canonical Dirac structure [58]. It induces the *implicit Lagrange–Poincaré equations* on the Lagrangian side and the *implicit Hamilton–Poincaré equations* on the Hamiltonian side. Note that this process of Dirac reduction does not only provide a geometric setting for the reduction of the equations of motion (both on the Lagrangian and Hamiltonian