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The sorting index on colored permutations and even-signed permutations ☆

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ABSTRACT

We define a new statistic sor on the set of colored permutations $\mathsf{G}_{r,n}$ and prove that it has the same distribution as the length function. For the set of restricted colored permutations corresponding to the arrangements of n non-attacking rooks on a fixed Ferrers shape we show that the following two sequences of set-valued statistics are joint equidistributed: $(\ell, \mathsf{Rmil}^0, \mathsf{Rmil}^1, \dots, \mathsf{Rmil}^{r-1}, \mathsf{Lmil}^0, \mathsf{Lmil}^1, \dots, \mathsf{Lmil}^{r-1}, \mathsf{Lmal}^0, \mathsf{Lmal}^1, \dots, \mathsf{Lmal}^{r-1}, \mathsf{Lmap}^0, \mathsf{Lmap}^1, \dots, \mathsf{Lmap}^{r-1})$ and $(\mathsf{sor}, \mathsf{Cyc}^0, \mathsf{Cyc}^{r-1}, \dots, \mathsf{Cyc}^1, \mathsf{Lmic}^0, \mathsf{Lmic}^{r-1}, \dots, \mathsf{Lmic}^1, \mathsf{Lmal}^0, \mathsf{Lmal}^1, \dots, \mathsf{Lmal}^{r-1}, \mathsf{Lmap}^0, \mathsf{Lmap}^1, \dots, \mathsf{Lmap}^{r-1})$. Analogous results are also obtained for Coxeter group of type D. Our work generalizes recent results of Petersen, Chen–Gong–Guo and Poznanović.

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1. Introduction

1.1. Mahonian and Stirling statistics

Let $\mathfrak{S}_n$ be the group of permutations on n letters $[n] := \{1, 2, \dots, n\}$. A pair (σ_i, σ_j) is called an *inversion* in a permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$ if $i > j$ and $\sigma_i < \sigma_j$. Denote by $\mathsf{inv}(\sigma)$ the number of inversions in σ. The distribution of inv over $\mathfrak{S}_n$ was first found by Rodriguez [9] to be

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathsf{inv}(\sigma)} = \prod_{i=1}^{n} [i]_q, \tag{1.1}$$

where $[i]_q := 1 + q + \cdots + q^{i-1}$.

In a Coxeter group, the *length* $\ell(\sigma)$ of a group element σ is the minimal number of generators needed to express σ. It is well known [2, Chapter 8] that $\mathfrak{S}_n$ is the Coxeter group of type A, where the generators are the adjacent transpositions and $\ell(\sigma) = \mathsf{inv}(\sigma)$. A permutation statistic is called *Mahonian* if it is equidistributed with inv over $\mathfrak{S}_n$. Similarly in a Coxeter group a statistic is called *Mahonian* if it is equidistributed with the length function ℓ.

The number of cycles cyc is another important statistic, whose distribution over $\mathfrak{S}_n$ is [10, Proposition 1.3.4]

$$\sum_{\sigma \in \mathfrak{S}_n} t^{\mathsf{cyc}(\sigma)} = \prod_{i=1}^{n} (t + i - 1). \tag{1.2}$$

As the coefficients of this polynomial are the (unsigned) Stirling numbers of the first kind, a permutation statistic over $\mathfrak{S}_n$ is called *Stirling* if it is equidistributed with cyc.

The *reflection length* $\ell'(\sigma)$ of σ in a Coxeter group is the minimal number of reflections (i.e., elements conjugate to generators) needed to express σ. In type A, the reflections are the transpositions and one has

$$\mathsf{cyc}(\sigma) = n - \ell'(\sigma). \tag{1.3}$$

1.2. Sorting index

Petersen [7] defined the *sorting index* sor over $\mathfrak{S}_n$ and proved it is Mahonian. One can uniquely decompose $\sigma \in \mathfrak{S}_n$ into a product of transpositions

$$\sigma = (i_1\, j_1)(i_2\, j_2) \cdots (i_k\, j_k)$$

with $j_1 < j_2 < \cdots < j_k$ and $i_1 < j_1, i_2 < j_2, \dots, i_k < j_k$. Then the sorting index of σ is