Advances in Applied Mathematics 61 (2014) 85–101

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Pattern avoidance in ordered set partitions and words ☆

Anisse Kasraoui

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria

ARTICLE INFO

Article history:
Received 22 July 2013
Received in revised form 14 July 2014
Accepted 29 August 2014
Available online 29 September 2014

MSC:
05A05
05A18
05A15
05A16

Keywords:
Permutation patterns
Ordered set partitions
Words
Exact enumeration
Asymptotic enumeration

ABSTRACT

We consider the enumeration of ordered set partitions avoiding a permutation pattern, as introduced by Godbole, Goyt, Herdan and Pudwell. Let $\mathrm{op}_{n,k}(p)$ be the number of ordered partitions of $\{1, 2, \ldots, n\}$ into k blocks that avoid a permutation pattern p. We establish an explicit identity between the number $\mathrm{op}_{n,k}(p)$ and the number of words avoiding the inverse of p. This identity allows us to easily translate results on pattern-avoiding words obtained in earlier works into equivalent results on pattern-avoiding ordered set partitions. In particular, (a) we determine the asymptotic growth rate of the sequence $(\mathrm{op}_{n,k}(p))_{n\geq 1}$ for every positive k and every permutation pattern p, (b) we partially confirm a conjecture of Godbole et al. concerning the variation of the sequences $(\mathrm{op}_{n,k}(p))_{1\leq k\leq n}$, (c) we undertake a detailed study of the number of ordered set partitions avoiding a permutation pattern of length 3. By the way, we observe that the number of words on k letters of length n avoiding a permutation pattern of length 3 can be written as a single sum by simplifying a formula of Burstein.

☆ Research supported by the grant S9607-N13 from Austrian Science Foundation FWF in the framework of the National Research Network "Analytic Combinatorics and Probabilistic Number Theory".

E-mail address: anisse.kasraoui@univie.ac.at.

http://dx.doi.org/10.1016/j.aam.2014.08.004
0196-8858/

1. Introduction

1.1. This paper is concerned with the enumeration of pattern-avoiding ordered set partitions. The subject, which can be seen as a generalization of the study of pattern-avoiding permutations, was initiated by Godbole, Goyt, Herdan and Pudwell in [7]. Recall that an ordered partition of a set S is a sequence of nonempty and mutually disjoint subsets, called blocks, whose union is S. When S is a subset of integers, it is usual to separate the blocks by a slash and to arrange the elements within each block in increasing order. We let $\mathcal{OP}_{n,k}$ denote the set of ordered partitions of $[n] := \{1, 2, \dots, n\}$ into k blocks. For instance, $2\,7/3/1\,4\,8/5\,6$ and $3/2\,7/1\,4\,8/5\,6$ are two (distinct) members of $\mathcal{OP}_{8,4}$.

Godbole et al. [7] considered the following notion of pattern containment: an ordered partition $\pi = B_1/B_2/\cdots/B_k$ in $\mathcal{OP}_{n,k}$ is said to contain a permutation $p = p_1\,p_2\,\cdots\,p_m$ in $\mathfrak{S}_m$, the symmetric group of the set $[m]$, as a pattern if there exist integers $i_1, i_2, \dots, i_m$, with $1 \le i_1 < i_2 < \cdots < i_m \le k$, and there exists $b_j \in B_{i_j}$ such that $b_1\,b_2\,\cdots\,b_m$ is order isomorphic to p. Otherwise we say that π avoids p. For example, $2\,7/3/1\,4\,8/5\,6$ contains the pattern $p = 213$, as evidenced (for instance) by $b_1 = 2 \in B_1$, $b_2 = 1 \in B_3$ and $b_3 = 5 \in B_4$. The number of members of $\mathcal{OP}_{n,k}$ that avoid the pattern p will be denoted by $\mathrm{op}_{n,k}(p)$. The special case $k = n$ is of particular interest since we clearly have

$$\mathrm{op}_{n,n}(p) = s_n(p), \tag{1.1}$$

where $s_n(p)$ stands for the number of permutations in $\mathfrak{S}_n$ that avoid the pattern p in the usual sense (see e.g. [2, Chapter 4]).

Finding a closed formula for $\mathrm{op}_{n,k}(p)$ is in general a hopeless task. Obviously, we have $\mathrm{op}_{n,k}(1) = 0$ for all $n \ge k \ge 1$. It is also easy to show [7] that

$$\mathrm{op}_{n,k}(12) = \mathrm{op}_{n,k}(21) = \binom{n-1}{k-1} \quad (n \ge k \ge 1). \tag{1.2}$$

The first nontrivial case is that of patterns of length three. Godbole et al. proved that $\mathrm{op}_{n,k}(p)$, for any $n, k \ge 1$, is the same for all $p \in \mathfrak{S}_3$, i.e. $\mathrm{op}_{n,k}(p) = \mathrm{op}_{n,k}(321)$ for all $p \in \mathfrak{S}_3$. By (1.1), this nicely generalizes Knuth's observation (see e.g. [2, Chapter 4]) that the number of permutations of $[n]$ that avoid the pattern p is the same for all $p \in \mathfrak{S}_3$. Table 1 lists the first few values of $\mathrm{op}_{n,k}(321)$. That the first values of the diagonal $(\mathrm{op}_{n,n}(321))_{n\ge1}$ coincide with those of the Catalan sequence comes as no surprise because of Knuth's well-known result (see e.g. [2, Chapter 4]) that $s_n(p)$, for all $p \in \mathfrak{S}_3$, is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

The study of the array $[\mathrm{op}_{n,k}(321)]_{n\ge k\ge1}$, which has also been considered in [7,5], is one of the central themes in this paper with which Section 3 is wholly concerned. We will also derive interesting results for general pattern p. To this end, we first need to state