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Asymptotic variance of grey-scale surface area estimators

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ABSTRACT

Grey-scale local algorithms have been suggested as a fast way of estimating surface area from grey-scale digital images. Their asymptotic mean has already been described. In this paper, the asymptotic behavior of the variance is studied in isotropic and sufficiently smooth settings, resulting in a general asymptotic bound. For compact convex sets with nowhere vanishing Gaussian curvature, the asymptotics can be described more explicitly. As in the case of volume estimators, the variance is decomposed into a lattice sum and an oscillating term of at most the same magnitude.

1. Introduction

The motivation for this paper comes from digital image analysis. Scientists in e.g. materials science and neurobiology are analyzing digital output from microscopes and scanners in order to gain geometric information about materials [10,15]. Common features of interest are volume, surface area, and Euler characteristic, as well as curvature and

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anisotropy properties. The focus of this paper will be on surface area estimation. Convergent algorithms for surface area are known, but as the amount of output data is typically quite large, there is a need for faster algorithms.

The simplest model for a digital image is a black-and-white image. If $X \subseteq \mathbb{R}^d$ is the object under study, the set of black pixels is modeled by $X \cap \mathbb{L}$ where $\mathbb{L}$ is a lattice. It is well known that the volume of X can be estimated by $c_{\mathbb{L}} \cdot \#(X \cap \mathbb{L})$ where $c_{\mathbb{L}}$ is the volume of a lattice cell and $\#$ is cardinality. If $\mathbb{L}$ is randomly translated, the mean estimate is exactly the volume.

Surface area is often estimated in a similar way [10,12,15]. The idea is to count the number of times each of the 2^{n^d} possible $n \times \cdots \times n$ configurations of black and white points appear in the image and estimate the surface area by a weighted sum of configuration counts. The advantage of these so-called local algorithms is that the computation time is linear in the data amount, see [16]. However, they are generally biased, even when the resolution tends to infinity [21,25].

A more realistic model for a digital image is that of a grey-scale image where we do not observe the indicator function $\mathbb{1}_X$ for X itself on $\mathbb{L}$, but rather its convolution $\mathbb{1}_X * \rho$ with a point spread function (PSF) ρ. In [22], local algorithms for grey-scale images are suggested and these are shown to be asymptotically unbiased when the lattice is stationary random and the PSF becomes concentrated near 0. They resemble the volume estimators as they are also given by lattice point counting, but each lattice point must now be weighted according to its grey-value. A simple such algorithm is given by counting the number of lattice points with grey-value belonging to a fixed interval.

So far, not much is known about the precision of these algorithms. Even though the mean converges, the variance may be large. While the convergence of the mean is independent of resolution, a low resolution would intuitively result in a large variance. The purpose of this paper is to study the variance using theory developed for the volume case.

The first part of the paper provides an asymptotic bound on the variance when resolution and PSF changes. It shows that the biggest contribution comes from the resolution. The bound explicitly depends on the algorithm and the underlying PSF.

The asymptotic bound is rather abstract and thus not useful for applications. In the second part of the paper, more explicit formulas for the variance are derived. As in the volume case, this requires strong conditions on the underlying set, namely smoothness, convexity, and nowhere vanishing Gaussian curvature. As in the volume case [14], the variance can be decomposed into a lattice sum depending only on X through its surface area and an oscillating term of at most the same magnitude.

The model for grey-scale images is introduced in Subsection 2.1 and local estimators and the known results about their mean are described in Subsection 2.2. A short recap of some of the known results for volume estimators is included for comparison in Subsection 2.3 before the main results of the paper are described in Subsection 2.4. In Sections 3–6, the main results are formally stated and proved. The paper ends with a discussion of the results and a list of open questions.