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Balanced simplices

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An additive cellular automaton is a linear map on the set of infinite multidimensional arrays of elements in a finite cyclic group $\mathbb{Z}/m\mathbb{Z}$. In this paper, we consider simplices appearing in the orbits generated from arithmetic arrays by additive cellular automata. We prove that they are a source of balanced simplices, that are simplices containing all the elements of $\mathbb{Z}/m\mathbb{Z}$ with the same multiplicity. For any additive cellular automaton of dimension 1 or higher, the existence of infinitely many balanced simplices of $\mathbb{Z}/m\mathbb{Z}$ appearing in such orbits is shown, and this, for an infinite number of values m. The special case of the Pascal cellular automata, the cellular automata generating the Pascal simplices, that are a generalization of the Pascal triangle into arbitrary dimension, is studied in detail.

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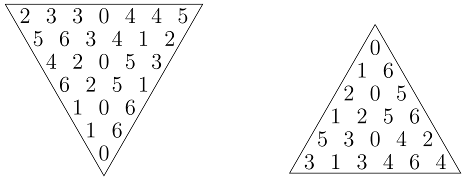

Fig. 1. A Steinhaus triangle and a generalized Pascal triangle modulo 7.

1. Introduction

Let m and s be two positive integers. A *Steinhaus triangle* modulo m of size s is a triangle, with s rows, whose first row is composed of s elements in $\mathbb{Z}/m\mathbb{Z}$ and which is satisfying the following local rule: each entry of the triangle (not on the first row) is the sum modulo m of the two directly above it. A *generalized Pascal triangle* modulo m of size s is defined like a Steinhaus triangle, but with the other possible orientation. More precisely, it is a triangle, with s rows, whose last row is composed of s elements in $\mathbb{Z}/m\mathbb{Z}$ and which is satisfying the same local rule. Examples of these triangles are depicted in Fig. 1.

A typical problem on such triangles is to determine sizes for which there exists a *balanced* triangle modulo m, that is a triangle containing all the elements of $\mathbb{Z}/m\mathbb{Z}$ with the same multiplicity. For instance, the triangles in Fig. 1 are balanced modulo 7.

The purpose of this paper is to study balanced simplices generated by additive cellular automata, that are a generalization of Steinhaus triangles and generalized Pascal triangles into arbitrary dimension and satisfying a local rule that is not necessarily the sum modulo m, but any linear map.

1.1. Known results on Steinhaus triangles and generalized Pascal triangles

The name of Steinhaus triangle is due to Hugo Steinhaus himself in [26], where he proposed this construction in the binary case $m = 2$. He posed the following problem, as an unsolved problem.

Problem 1.1. (See Steinhaus [26].) Does there exist, for all positive integers s such that $s \equiv 0$ or 3 mod 4, a Steinhaus triangle of size s in $\mathbb{Z}/2\mathbb{Z}$ containing as many 0's as 1's?

Remark that the condition on the size s of a balanced Steinhaus triangle in $\mathbb{Z}/2\mathbb{Z}$ is obviously a necessary condition because the number of elements of a such triangle, that is $\binom{s+1}{2}$, is even if and only if $s \equiv 0$ or 3 mod 4. A positive solution to this problem appeared in the literature for the first time in [23], where the author gave, for every $s \equiv 0$ or 3 mod 4, an explicit construction of a balanced Steinhaus triangle of size s in $\mathbb{Z}/2\mathbb{Z}$. More recently, several other constructions of balanced binary Steinhaus triangles have been obtained by considering sequences with additional properties such as strongly balanced [20], symmetric and antisymmetric [21], or zero-sum sequences [22].